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Hierarchical Bayesian fatigue data analysis

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ABSTRACT

The problem minimizing the number of specimens required for fatigue data analysis is considered in this research. Assuming unknown hyperparameters described via prior distributions, a hierarchical Bayesian model with accumulated prior information was proposed to deal with this issue. One of the main advantages of hierarchical Bayesian model over the empirical Bayesian model is that the prior distributions with hierarchical structure can incorporate structural prior and subjective prior simultaneously. The probabilistic stress-cycle (*P-S-N*) curves are generated from the predictive distributions, involving both the randomness of parameters and the scatter of observations, and calculated by an identical hierarchical structure. The numerical calculation is done via the Gibbs sampler, which makes the whole process simple and intuitive.

1. Introduction

The laboratory fatigue tests data, always displaying a large scatter observation, are often presented in the form of a median stress-cycle (*S-N*) curve in high cycle fatigue life prediction with a series of stress amplitudes, while *P-S-N* curves are used for studying the fatigue life reliability. There is no doubt that the estimation of unknown parameters of fatigue design curves relies heavily on statistical analysis. Miner [1] suggested a cumulative damage rule in 1945, as the well-known Palmgren-Miner's rule:

$$D = \sum_{i}^{n} \frac{n_i}{N_i} \quad \text{for } i = 1, \ldots, n. \tag{1}$$

Eq. (1) describes a linear damage rule, D is the cumulative damage of n_i cycles at each stress level S_i, and N_i is the number of cycles to failure. The number of stress cycles of a certain material is determined experimentally by applying constant or variable amplitude load and recording the number of cycles to failure. N_i is usually assumed to depend on the load amplitude S_i by Basquin relation [2]

$$N_i = AS_i^{-B} \quad \text{for } i = 1, \ldots, n. \tag{2}$$

where $A > 0, B > 0$ are fatigue curve coefficients. In a more complex three parameters model [3],

$$N_i = A(S_i - S_0)^{-B} \quad \text{for } i = 1, \ldots, n. \tag{3}$$

where S_0 is a material constant too, which can be the mean life of the specimen [4] or the fatigue limit [5]. Random effects of the mechanical properties of structure and material, and even environmental factors lead to scattered fatigue life cycle data. The fatigue lives of similar specimens or structures under the same fatigue load can be significantly different. Some research [6,7] proposed to utilize the regularized fatigue life to deal with the random nature of the fatigue life data and to examine the continuous dependence of the fatigue life curve on the measured data. In order to describe material fatigue precisely, fatigue life dispersion characteristics usually are expressed by *P-S-N* curves. In fatigue reliability analysis, Eq. (2) can be extended to represent *P-S-N* curves at survival probability p as

$$N_{ip} = A_p S_i^{-B_p} \quad \text{for } i = 1, \ldots, n. \tag{4}$$

Similarly, in the case of three parameters,

$$N_{ip} = A_p(S_i - S_{0p})^{-B_p} \quad \text{for } i = 1, \ldots, n. \tag{5}$$

where A_p, B_p and S_{0p} are material constants, and N_{ip} is the life corresponding to survival probability p at the stress level S_i. The scatter of material parameters can be expressed by the lognormal distribution or a three parameters Weibull distribution, stated by Weibull [8] in 1939. A zero fatigue life of the lognormal distribution is physically impossible, so this discrepancy does not occur in the three parameters Weibull distribution function. In fact, the lognormal distribution can fit fatigue data well at the high or low stress levels, while Weibull distribution achieves better results at medium stress

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levels [9]. Unfortunately, the distribution function of fatigue life hardly can be derived on the basis of physical arguments. In a reason, the validation of the two above distribution functions requires large test data [10], which is time-consuming and costly.

Research works have been carried out for *P-S-N* curves estimation from small data sets, often involving complex data structures, by statistical methods, such as maximum likelihood method with a reference stress level [11], empirical Bayesian model [12], the random fatigue limit model [5], backwards statistical inference method [13], etc., driven by data analysis or physical models. The notional fatigue reliability on the design curve only reflects the observed physical uncertainty associated with the fatigue process itself, but for the small fatigue data set the statistical uncertainties concerning the point parameter estimates could be equally significant. Bayesian analysis has been presented for establishing design *S-N* curves from small censored data sets to solve underlying statistical uncertainties [14]. Although a number of papers have appeared which exploited Bayesian inference in the analysis of the propagation of fatigue cracks, very few attempts have been made in the past to use the Bayesian approach in the context of *S-N* fatigue tests [12]. The Bayesian inference approach focuses on updating the probability for a hypothesis of the parameters on the basis on observations [15]. Bayesian linear regression has been considered in fatigue data analysis where the posterior distribution of model parameters was then used to predict the fatigue life [16]. The empirical prior knowledge was derived from material parameters [12], maximum likelihood estimates [17], or test series with similar components of different geometry [16]. Different residual stress and strain data measured from various techniques are analyzed using a Bayesian statistical approach and then interpolated utilizing modified Shepard method [18], which has been used for fatigue life measurements analysis indirectly [6]. All of the aforementioned Bayesian inference applications of fatigue data analysis focused on using empirical Bayesian model, while hierarchical Bayesian models with structural and subjective hierarchical priors, both or one of them, have been applied to describe the fatigue crack growth [19] or the crack growth rate [20].

The present research primary focuses on the fatigue curves estimation method by using hierarchical Bayesian model, for both the two parameters and three parameters methods. Then the predictive distributions in the same hierarchical structure are proposed to estimate the *P-S-N* curves. Moreover, one of the main advantages of the hierarchical Bayesian model is that it allows the use of both structural prior information and subjective prior information simultaneously. As a result, the choice of noninformative prior distributions is discussed in detail, as well as the model checking and the convergence of Gibbs sampling. Finally, numerical examples of hierarchical Bayesian models for estimating the *S-N* and *P-S-N* curves from the collection of real data under study are presented, along with their comparisons of the maximum likelihood estimation (MLE).

2. Hierarchical Bayesian models for estimating *S-N* curves

2.1. Fundamentals of hierarchical Bayesian models

From the perspective of Bayesian statistics, the parameters within models are regarded as random variables, and thereby having probability distributions, which are known as prior distributions. An important type of prior distribution is a hierarchical prior, since it is often convenient to model structural knowledge in stages. Distinguishing feature of the hierarchical Bayesian approach to empirical Bayesian analysis is the hierarchical nature in which information is accumulated. As a result, the hierarchical approach is most commonly used approach to building complex models by specifying a series of more simple conditional distributions [21]. Generally, hierarchical models are more flexible than the typical nonhierarchical models since a more complicated structure is accommodated in the model. After ignoring the normalizing constant $f(\boldsymbol{y})$, a hierarchical Bayesian model is defined by

$$f(\boldsymbol{\upsilon}, \boldsymbol{\theta}|\boldsymbol{y}) \propto f(\boldsymbol{y}|\boldsymbol{\theta}, \boldsymbol{\upsilon})f(\boldsymbol{\theta}|\boldsymbol{\upsilon})f(\boldsymbol{\upsilon}) \tag{6}$$

with a first stage likelihood, $f(\boldsymbol{y}|\boldsymbol{\theta}, \boldsymbol{\upsilon})$, and second stage density, $f(\boldsymbol{\theta}|\boldsymbol{\upsilon})$. $\boldsymbol{\theta}$ and $\boldsymbol{\upsilon}$ represent likelihood parameters and hyperparameters of prior distributions respectively, and $f(\cdot)$ are distribution functions.

Comparing with the empirical Bayesian model, for which the prior distribution is fixed before any data are observed, the hierarchical Bayesian model offers many advantages [21–23]. Empirical Bayesian model is failure to consider hyperparameter estimation error and also does not indicate how to incorporate the hyperparameter estimation error in the analysis by itself, while the hierarchical Bayesian analysis incorporates such errors automatically. Another advantage is that the hierarchical structure leads to a more robust analysis, since it reduces the arbitrariness of the hyperparameter choice. Moreover, the hierarchical Bayesian model can incorporate actual subjective prior information at the second stage, so that it allows the use of both structural prior information and subjective prior information simultaneously, random fatigue limit model in fatigue data analysis (Section 2.2), for example. Finally, empirical Bayes theory requires the solution of likelihood equations, while the hierarchical Bayes approach requires numerical integration, Markov Chain Monte Carlo (MCMC) algorithms for instance, and resulting in conditional distributions. The increasing applications and practical implementations of Bayesian models have owed much to the development of MCMC algorithms, such as Gibbs sampling [24], Metropolis-Hastings algorithm, etc. for estimation [25,26] relatively recently. The Gibbs sampler, also called alternating conditional sampling, the proposal density of which can be generated by iteratively sampling from the full conditional posterior distribution $p_{\theta|y}(\theta_j|\boldsymbol{\theta}_{\setminus j}, \boldsymbol{y})$, where $\boldsymbol{\theta}_{\setminus j} = (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_d)^T$. As a special case of the Metropolis-Hastings algorithm, the Gibbs sampler meets the detailed balance condition and leads to a stationary distribution, but samples one random variable at a time and does not require the check for the proposed sample acceptance.

The Bayesian approach expands the class of models to fit data, enabling one to handle complex correlations, unbalanced or missing data, etc. In addition to the advantages described above, the hierarchical Bayesian models, in a sense, all Bayesian models are hierarchical [15], provide a formal framework for analysis with a complexity of structure that matches the system being studied.

2.2. Models for estimating S-N curves

If the random error is taken into consideration, Eq. (2) can be expressed as:

$$N_i = AS_i^{-B}\varepsilon_i \tag{7}$$

where ε_i indicates the randomness in stress level S_i, comprising the random effect of materials and random error in observations. Take the natural logarithm transformation for Eq. (7),

$$\log_{10}N_i = \log_{10}A - B\log_{10}S_i + \delta_i \tag{8}$$

The random variables δ_i in Eq. (8) can be a normal distribution, Weibull distribution, or Gaussian mixture distribution, of which the mixing measure, which uses a unfixed number of parameters or accounts for uncertainty about distributional shape [27,28], can be a Dirichlet process in a Bayesian nonparametric approach. For the purpose of introducing the hierarchical Bayesian models in this paper, the normal distribution is chosen in the subsequent study.

Therefore, we assume $\delta_i = \sigma_i e_i, \sigma_i$ is the standard deviation of the logarithmic fatigue life under the stress level S_i and $e_i \sim \mathcal{N}(0,1)$. Following Guida's [12] suggestion, let $u_i = \log_{10} S_i$ and $\bar{u} = (1/n)\sum_i^n \log_{10} S_i$, then

$$y_i = \alpha + \beta x_i + \mathcal{N}(0, \sigma_i^2) \tag{9}$$

where $y_i = \log_{10} N_i, x_i = u_i - \bar{u}, \alpha = \log_{10} A - B\bar{u}$ and $\beta = -B$. For statistical analysis it is convenient to rewrite the Eq. (9) to a normal distribution with mean $\mu = \alpha + \beta x_i$ and variance σ_i^2 [12]. which can be expressed as:

$$y_{ij} \sim \mathcal{N}(\alpha + \beta x_i, \sigma_i^2) \tag{10}$$

or

$$y_{ij} \sim \mathcal{N}(\mu_i, \sigma_i^2) \quad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m. \tag{11}$$

where $\mu_i = \alpha + \beta x_i, y_{ij}$ is the jth observation of the ith stress level. The prior distributions in the second stage are assumed that $\alpha|\mu_\alpha, \sigma_\alpha \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2), \beta|\mu_\beta, \sigma_\beta \sim \mathcal{N}(\mu_\beta, \sigma_\beta^2)$ (see Section 4.1 for more information on the choice of prior distributions). The directed acyclic graph (DAG) of the hierarchical sturcture is shown in Fig. 1. In this representation, variables are arranged in a series of levels, with data in the innermost and hyperparameters in the outermost. The arrows represent dependencies of variables, which are assumed to be independent conditional on each level. Known data are placed in the box and unknown variables are put in the circles. Similarly, the three parameters expression for S-N curves is assumed as

$$N_i = A(S_i - S_0)^{-B} \varepsilon_i \tag{12}$$

where ε_i is also assumed to be distributed as a log-normal random variable, same as the two parameters method:

$$\log_{10} N_i = \log_{10} A - B\log_{10}(S_i - S_0) + \sigma_i e_i \tag{13}$$

Let $\alpha = \log_{10} A$ and $\beta = -B$, then

$$y_i = \alpha + \beta\log_{10}(x_i - \gamma) + \sigma_i e_i \tag{14}$$

where $x_i = S_i, \gamma = S_0$, and

$$y_{ij} \sim \mathcal{N}(\alpha + \beta\log_{10}(x_i - \gamma), \sigma_i^2) \tag{15}$$

or

$$y_{ij} \sim \mathcal{N}(\mu_i, \sigma_i^2) \quad \text{for } i = 1, \ldots, n \text{ and } j = 1, \ldots, m. \tag{16}$$

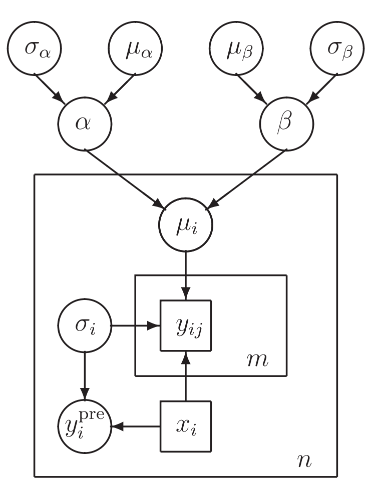


Fig. 1. The DAG of the hierarchical Bayesian model for estimating S-N curve or P-S-N curves. The arrows represent dependencies of variables, which are assumed to be independent conditional on each level. These nodes with fixed quantities in the analysis are represented by rectangles, while unobserved random variables are represented by circles.

where $\mu_i = \alpha + \beta\log_{10}(x_i - \gamma), y_{ij}$ is the jth observation of the ith stress level. Note that the x_i in the three parameters model is very different from the two parameters model. $\alpha|\mu_\alpha, \sigma_\alpha \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$, $\beta|\mu_\beta, \sigma_\beta \sim \mathcal{N}(\mu_\beta, \sigma_\beta^2), \gamma|\mu_\gamma, \sigma_\gamma \sim \mathcal{N}(\mu_\gamma, \sigma_\gamma^2)$ are still assumed in the second stage. Suppose that, γ is the fatigue limit of the specimen, and $f_V(v; \mu_\gamma, \sigma_\gamma)$ is probability density function of V by letting $V = \log(\gamma)$

$$f_V(v; \mu_\gamma, \sigma_\gamma) = \frac{1}{\sigma_\gamma}\phi V\left(\frac{v - \mu_\gamma}{\sigma_\gamma}\right) \tag{17}$$

where $\phi V(\cdot)$ is either the standardized smallest extreme value or normal probability density function. When $f_V(v; \mu_\gamma, \sigma_\gamma)$ is the prior of γ, that is $V \sim f_V(v; \mu_\gamma, \sigma_\gamma)$, then Eq. (15) can be considered as a kind of random fatigue-limit model [5].

3. *P-S-N* curves estimation by using predictive distributions

The fatigue P-S-N curve with the given survival probability p is easily generated via the parameters α_p and β_p. Eq. (4) can represent P-S-N curve at survival probability p as

$$\log_{10} N_{ip} = \log_{10} A_p - B_p \log_{10} S_i \tag{18}$$

The logarithm of fatigue life is assumed to normally distribute previously, so that the logarithm of fatigue life at survival probability $p, \log_{10} N_{ip}$, can be expressed as

$$\log_{10} N_{ip} = \log_{10} N_i - \mu_p \sigma_i \tag{19}$$

where μ_p is the standard normal deviate corresponding to survival probability p. Substituting Eqs. (8) and (18) into Eq. (19) yields

$$\mu_p \sigma_i = \alpha - \alpha_p + (\beta - \beta_p)x_i \tag{20}$$

where $\alpha_p = \log_{10} A_p - B_p\bar{u}$ and $\beta_p = -B_p$. However, the model complexity and computational cost are greatly increased, therefore another approach will be discussed to obtain the P-S-N curves using prediction of observations $\boldsymbol{y}$ in Section 3.2 within the same hierarchical structure.

3.1. The predictive distributions

The predictive distributions of observations can be presented within the same hierarchical structure. The Gibbs sampler takes advantages of hierarchical structures [23], when a Bayesian model can be written as

$$f_y(\boldsymbol{y}^{\mathrm{pred}}) = \int f_{y|\theta}(\boldsymbol{y}^{\mathrm{pred}}|\boldsymbol{\theta}) f_\theta(\boldsymbol{\theta}) d\boldsymbol{\theta} \tag{21}$$

where $f_y(\cdot)$ are predictive distributions or prior predictive distributions. It is easy to generate the prediction $\boldsymbol{y}^{\mathrm{pred}}$ by using the predictive distributions. The $f_y(\cdot)$ is the stationary distribution of this Markov chain [29], expressed as follows:

$$f_y(\boldsymbol{y}^{\mathrm{pred}}) = \int \mathcal{K}(\boldsymbol{y}^{\mathrm{pred}}|\boldsymbol{y}) f_y(\boldsymbol{y}) d\boldsymbol{y} \tag{22}$$

where $\mathcal{K}(\boldsymbol{y}^{\mathrm{pred}}|\boldsymbol{y})$ is the transition kernel for the chains of $\boldsymbol{y}^{\mathrm{pred}}$, given by:

$$\mathcal{K}(\boldsymbol{y}^{\mathrm{pred}}|\boldsymbol{y}) = \int f_{y|\theta}(\boldsymbol{y}^{\mathrm{pred}}|\boldsymbol{\theta}, \boldsymbol{y}) f_{\theta|y}(\boldsymbol{\theta}|\boldsymbol{y}) d\boldsymbol{\theta} \tag{23}$$

3.2. P-S-N curves estimation by using predictive distributions

The fatigue life cycles $\boldsymbol{y}^{\mathrm{pred}}$ of all the stress levels can be easily generated from the posterior predictive distributions by adding a single simple step within the Gibbs sampler, that is the expected life values in all stress levels $E(\boldsymbol{y}^{pred}|\boldsymbol{y}, \boldsymbol{x})$ can be estimated by using the predictive distributions

$$f(\boldsymbol{y}^{pred}|\boldsymbol{y}) = \int f(\boldsymbol{y}^{pred}|\boldsymbol{\mu}, \boldsymbol{\sigma}, \boldsymbol{y}) f(\boldsymbol{\mu}, \boldsymbol{\sigma}|\boldsymbol{y}) d\boldsymbol{\mu} d\boldsymbol{\sigma} \tag{24}$$

where $\boldsymbol{\sigma}$ is the standard deviation, $\boldsymbol{\mu} = \alpha + \beta\boldsymbol{x}$, the same as previously mentioned, $\boldsymbol{y}^{pred}$ denotes the future data for predicting, and $\boldsymbol{y}$ is the complete test data. When the observations of some stress levels is insufficient, the observations of fatigue life can be treated as incomplete data, and the insufficient data is considered as missing data. That is, the complementary data $\boldsymbol{y}$ include the observations and the missing data $\boldsymbol{y}^{miss}$ in this case. The survival function $S(y_i) = 1 - F(y_i) = 1 - P(Y_i \leqslant y_i)$, the probability of surviving at the fatigue life y_i at stress level S_i, can be expressed using the predictive distributions naturally, so that the *P-S-N* curves can be obtained by using prediction of observations $\boldsymbol{y}$ easily.

Follow the discussion in Section 2.2, the same effect as Ling's maximum likelihood method [11] can be achieved by making $\sigma_i = \sigma_r$ in Eqs. (11) and (15) at each stress level S_i, where σ_r is the various standard deviations of fatigue life at reference stress, i.e., choosing a stress level S_r as a reference stress level and a group of specimens are tested. Note that the reference stress used here is very different from the reference prior in the objective Bayesian. It will greatly reduce the number of test samples when the reference stress is used, consistent with the maximum likelihood method, of course, the same limitation too. It is based on the assumption that the statistical nature of the life data is the same at all applied stress levels of interest. In the Bayesian hierarchical model, an unknown σ can be assumed to characterize the dispersion of fatigue life at each stress level, that is $\sigma_i = \sigma$, without providing a reference stress level. It might be more reasonable because the fatigue life cycle scattered nature of the stress levels of all available data are taken into account.

All of the above discussions are based on the assumptions that there are similarities in the distributions of fatigue life at different stress levels. An assumption of "similarity" here is not equivalent to the assumption of "exchangeability", which was shown to be equivalent to assuming that the observations of each stress level were independent and identically distributed from one distribution with the hyperparameters unknown, and the priors of those hyperparameters are given. These hyperparameters should arise from a common "population" distribution whose parameters are unknown and assigned appropriate prior distributions [30]. Hierarchical models tend to recognize that it is unlikely that all stress levels have the same underlying survival rate by giving different distribution parameter σ_i, with the same prior distribution, for the various stress levels.

In addition, in dealing with the *S-N* or *P-S-N* estimate problem, the fatigue measured data often have missing data problems or there are not as many observations at different stress levels. Less data means less expense and test time, especially at lower stress levels. The hierarchical Bayesian model can easily deal with these situations by using the predictive distributions since the missing y_i^{miss} of x_i can be considered as an additional parameter under estimation. Obviously, more missing data also means more uncertainty in the estimation and therefore results in more conservative estimates.

4. Prior choice and model checking

4.1. The choice of prior distributions

The first step of Bayesian inference is to determine the prior distributions of parameters $\boldsymbol{\theta}$. Undoubtedly, one of the most critical and most criticized points of Bayesian analysis is that it deals with the choice of the prior distribution, so that the prior distribution is the key to Bayesian inference and its determination is, therefore, the most important step in drawing this inference. There are two kinds of prior, the informative prior and the noninformative prior. The informative prior could be based upon some or all of the following: (1) mathematical or physical models, (2) engineering, physics, etc. information, (3) expert's judgements, (4) historical data of the same or similar circumstances, and other reasonable information.

The noninformative prior can be directly derived from the sampling distribution. Jeffreys [31] described a method to derive the prior distribution directly from the sampling distribution. For normal distribution, consider $y \sim N(\mu, \sigma^2)$, with μ, σ unknown. In the Jeffreys noninformative prior case, the corresponding noninformative prior distribution is $\pi \propto 1/\sigma^2$. $\mathcal{G}(\epsilon, \epsilon)$, with ϵ small and positive, is 'just proper' form of Jeffreys prior [32]. In the particular case of conjugate distributions in the usual exponential families, the posterior expectations of the natural parameters can obviously be expressed analytically. The normal distribution, for example, the conjugate prior of its mean is still normal distribution, while the variance of it has an inverse gamma conjugate prior [23]. If the prior information is weak, $\theta_i \sim \mathcal{N}(\mu_{\theta_i}, \sigma^2_{\theta_i})$, in which $\mu_{\theta_i}, \sigma_{\theta_i}$ are hyperparameters, can be chosen, and μ_{θ_i} is often set to 0, while σ_{θ_i} is often set to 10^k with a sufficiently large k in Bayesian model [33]. However, $\theta_i \sim \mathcal{N}(\mu_{\theta_i}, \sigma^2_{\theta_i})$ is set in hierarchical Bayesian model, and the hyperparameter θ_i are assumed to have noninformative or weakly-informative prior distributions. In addition, to assess sensitivity to prior assumptions, the analysis may be repeated over a limited range of alternative priors. Some researchers [34,35] suggested a gamma prior on inverse variance, $1/\sigma^2$, governing random walk effects (e.g., baseline hazard rates in survival analysis), namely, $1/\sigma^2 \sim \mathcal{G}(a, b)$, where a is set at 1, but b is varied over choices such as 0.05 or 0.0005. Unfortunately, Gelman [32] shows that inferences become very sensitive to a, b, especially for problems where the group-level variance σ_{θ_i} is close to zero and is crucially affected by the choice the scale of σ_{θ_i}. Thus a, b must be set to a reasonable value, so that the prior distribution hardly looks noninformative. Gelman also recommended starting with a noninformative uniform prior density $\mathcal{U}(0, u)$ on a wide range of standard deviation parameters in fitting hierarchical models. As a result, uniform distributions and the inverse gamma distributions are selected as a prior distribution in Section 5.

The physical model can not only be expressed as a likelihood function, but also, as described in Section 2.2, can be incorporated into a hierarchical Bayesian model as an informative prior conveniently. Moreover, Statistical uncertainties of historical data can also be easily added to the hierarchical Bayesian model as an informative prior. In fact, for a large sample size, MLE of a scalar parameter θ, say $\hat{\theta}$, is approximately normally distributed with mean θ and variance equal to the negative reciprocal of the observed information, $I(\theta)$. Similarly, because the weight given to the prior mean decreases to 0 as the number of experimental becomes large, the posterior distribution will converge to a normal distribution centered on the MLE, and the variance of the posterior distribution converges to the inverse of the $I(\theta)$. Therefore the asymptotic (large sample) properties of the MLE and the posterior distribution are similar in this sense [33]. As a result, it is reasonable that the asymptotic distribution is selected as an informative prior distribution.

4.2. Convergence of the MCMC algorithm and model checking

As discussed in Section 2.1, the target posterior distributions and the hyperparameters noninformative prior distributions are easily obtained by Gibbs sampling. MCMC methods broadly applicable, but require care in parametrization and convergence diagnosis. In order to obtain correct target posterior distributions, there are many ways to monitor convergence, including the MC error (the simplest way), the trace plots, the ergodic mean, plotting autocorrelations etc. Multiple-chain comparisons with different initial values are also efficient in practice, and BrooksGelmanRubin (BGR) diagnostics [36] also can be employed in this case.

The deviance information criterion (DIC), which can be applied to compare the fatigue estimation models, is obtainable as the expected deviance plus the effective model dimension, and was introduced by Spiegelhalter et al. [37]. The DIC can be seen as a Bayesian version of Akaike information criterion (AIC),

$$\mathrm{DIC} = 2\overline{D} - D(\bar{\theta}) = D(\bar{\theta}) + 2p_D \tag{25}$$

where $D(\bar{\theta})$ is the usual deviance measure, which is equal to minus twice the log-likelihood $D(\bar{\theta}) = -2\log f(y|\theta)$ and $\overline{D}$ is its posterior mean, p_D can be interpreted as the number of effective parameters given by $p_D = \overline{D} - D(\bar{\theta})$ and $\bar{\theta}$ is the posterior mean of the parameters. A probability density $f(y|\theta)$ can be greater than 1 for a small standard deviation, hence a deviance can be negative, and then generate a negative DIC. In any case, smaller DIC values always indicate a better-fitting model. In fact, Kelly proposed using BIC of non-hierarchical models while prefering DIC if hierarchical models are used [38].

For hierarchical models, expectation posterior predictive p-values can be calculated [39,40] under current posterior distribution by new data $\boldsymbol{y}^{\text{new}}$. Equivalently, the p-values can be estimated by posterior probability that $P(\boldsymbol{y}^{\text{pred}} < \boldsymbol{y}^{\text{new}}), \boldsymbol{y}^{\text{pred}}$ is the predictive observation, which was described in detail in Section 3. However, it is questionable for model checking because of the repeated use of data. As a result, several researchers [41–44] have proposed the use of cross-validation predictive densities [22]. Geisser et al. [45] proposed using the leave-one-out cross-validation predictive density

$$f(y_i|\boldsymbol{y}_{\backslash i}) = \int f(y_i|\boldsymbol{\theta})f(\boldsymbol{\theta}|\boldsymbol{y}_{\backslash i})d\boldsymbol{\theta} \tag{26}$$

where $\boldsymbol{y}_{\backslash i}$ is $\boldsymbol{y}$ after omitting y_i.

5. Numerical examples

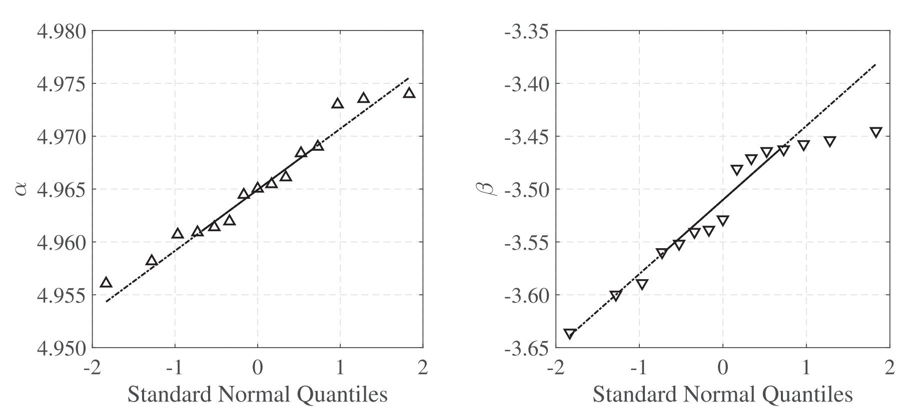
In order to verify the hierarchical Bayesian model in the fatigue data analysis, two numerical examples are carried out to assess hierarchical Bayesian model performances with respect to the MLE, and each example is discussed in two cases: the variances of different stress levels are same in Case 1 and the variances are different but they distribute from a same prior in Case 2, as described in Section 3.2. The data used in these examples are from the published literature [13,46]. For Example 1, fatigue test was conducted with standard plate specimens of alloy 2524-T3 under four stress levels with about 15 observations each by Xie et al. [13], see Table B.1. Since the sequence of the selected specimen's life data randomly, the asymptotic result of the parameters can be calculated, as shown in Fig. 2(a) and (b). The asymptotic distributions of parameters can be seen as normal distributions due to the small-scale of vertical axis, so that it is reasonable to choose normal distributions as prior distributions for parameters α and β. For Example 2, durability data for flat specimens cut from the S420MC steel plate is given by Klemenc et al. [46], and these standard specimens had a standard shape in accordance with the ASTM E606-92 standard. In these fatigue tests, 65 specimens were broken before 2 million load cycles but 15 specimens survived. These broken specimens were used to estimate the S-N curve and its scatter, see Table B.2, and they are also employed for cross-validation here.

5.1. Technical details

Gibbs sampling is employed to simulate previously mentioned distributions. WinBUGS [47] is used to achieve the purpose of Gibbs sampling in this paper, and all computations are done on a desktop PC. Note that the initial values given in these numerical examples, for both cases of the hierarchical Bayesian model and the MLE, are identical. The Gibbs sampling of two parameters case steps for the distributions are as follows:

(1) Specifying the likelihood $y_{ij} \sim \mathcal{N}(\mu_i, \sigma_i^2)$, where $\mu_i = \alpha + \beta x_i$;
(2) Sample α, β, given the other parameters, from $\mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$ and $\mathcal{N}(\mu_\beta, \sigma_\beta^2)$ respectively;
(3) Sample μ_α, μ_β from $\mathcal{N}(0, 10^6)$ and sample τ_α, τ_β from $\mathcal{G}(0.001, 0.001)$, Sample τ and τ_i from $\mathcal{G}(0.001, 0.001)$ in Case 1 and Case 2 respectively;
(4) For prediction or P-S-N curves estimation, sample y_i^{pred} from predictive distribution $\mathcal{N}(\mu_{y_i^{\text{pred}}}, \sigma^2_{y_i^{\text{pred}}})$, where $\mu_{y_i^{\text{pred}}} = \hat{\alpha} + \hat{\beta}x_i, \sigma_{y_i^{\text{pred}}} = \hat{\sigma}$ in Case 1 and $\sigma_{y_i^{\text{pred}}} = \hat{\sigma}_i$ in Case 2.

where $\tau_* = 1/\sigma_*^2$ for using inverse gamma distributions, and $*$ denotes subscripts. Table 1 shows the settings of prior distributions which are used for hierarchical Bayesian model to estimate the curves in these examples. In order to verify the convergence of MCMC, two chains with different initial values are used in the examples, both of which burnin period are 1000 iterations and then sample 10,000 times for each chain. Recalling the material



(a) Asymptotic distribution of α. (b) Asymptotic distribution of β.

Fig. 2. QQ plot of parameters to be estimated versus standard normal, the asymptotic distributions of parameters can be seen as normal distributions for the small-scale of vertical axis.

Table 1
The settings of prior distributions in Case 1.

	Second stage	Third stage
$\mu \vert \alpha, \beta$	$\alpha \vert \mu_\alpha, \tau_\alpha \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha^2)$	$\mu_\alpha \sim \mathcal{N}(0, 10^6)$ $\tau_\alpha \sim \mathcal{G}(0.001, 0.001)$
	$\beta \vert \mu_\beta, \tau_\beta \sim \mathcal{N}(\mu_\beta, \sigma_\beta^2)$	$\mu_\beta \sim \mathcal{N}(0, 10^6)$ $\tau_\beta \sim \mathcal{G}(0.001, 0.001)$
	$\tau,\ \tau_i \sim \mathcal{G}(0.001, 0.001)$	–

parameter $B > 0$, for the two parameters model, the initial values of one of the chains are set as follows: $\mu_\beta = \beta = -B = -1, \tau_i$ and y_i^{pred} are given a value close to the observed values y_{ij}, and the remaining variables are set to 1. The other group of initial values is completely arbitrary and all of which are set to 1 here. Concerning the sensitivity of the prior of hyperparameter σ_{θ_i} to inferences, the hierarchical model is calculated by setting $\tau_{\theta_i} \sim \text{U}(0, 10^4)$, and obtain a consistent result.

Since Example 2 is also employed for cross-validation, so that fatigue test data is divided into two parts as shown in Fig. 4. The data used for model estimation are divided into 5 groups, by putting the data of close stress levels together. Finally, Example 1 contains four sets of data, each 15, while Example 2 contains five sets, each 12, in both of which the lack of data is set as missing data.

5.2. Results and discussions

The convergence of MCMC in Example 1 is monitored as shown in Fig. A.1. Fig. A.1(a) and (d) give smooth density curves of the material parameters α and β. A low degree of autocorrelation in the sample as shown in Fig. fig:MCMC-convergence(b) and (e), BGR diagnostics are shown in Fig. fig:MCMC-convergence(c) and (f), in which the dashed line denotes the reference value of one, and as the number of iterations increase, the evolution of the pooled posterior variance (in green) and the mean of the variances within each sample tends to stabilization, and their ratio (in red) tends to one. All of these monitors, including the MC error results in Table 2, indicate the good convergence of the algorithm in Example 1.

For Case 1, with common variance, the results of Example 1 and Example 2 are given in Tables 2 and 3 respectively, while the MLE is calculated with results $\alpha = 4.9740, \beta = -3.4708$ of Example 1 and $\alpha = 5.5723, \beta = -8.0075$, of Example 2. The hierarchical Bayesian model fitting results of the examples are shown in Figs. 3 and 4, with 97.5% survival probability or 95% prediction intervals. Credible intervals, which are used for interval estimation, are intervals in the domain of the posterior probability distribution in Bayesian statistics, and they are analogous to confidence intervals in frequentist statistics. Although the predictive distribution is a kind of posterior distribution, but the prediction intervals here refers the predictive distribution domain of observations specifically. Obviously, the infimum of the 95% prediction interval corresponds to the 97.5% survival probability. The hierarchical Bayesian model with common variance (Case 1) results are also compared with the MLE, as shown in Figs. 7 and 8, and the hierarchical Bayesian model with independent variance (Case 2), as shown in Figs. 5 and 6, respectively. Figs. 5 and 6 illustrate that there is little difference between the estimation results of the one with common variance σ^2 and the one with several different variances σ_i^2, so it means that good enough results are obtained by the model of Case 1. The cross-validation is used for model checking in Example 2, the results show that the hierarchical Bayesian model used for fatigue data estimation is credible.

Table 4 shows a model comparison of between Case 1 and Case 2 via DIC values, for Example 1 and 2 respectively. The negative DIC values is easy to understand for the logarithm of the data narrow the scale, resulting in a small variance. As Section 4.2 discussed, smaller DIC values indicate a better-fitting model. Case 2 in Example 1 and Case 1 in Example 2 have smaller DIC and achieve better fitting models. Corresponding to the distributed characteristics of the data, the different variances σ_i^2 settings prefer to suit large differences in distribution of life in different stress levels, while a common variance σ^2 setting for small difference is recommended.

Compared with the MLE, the advantage of the hierarchical Bayesian model for the *S-N* curve fitting is inconspicuous, while the hierarchical Bayesian approach highlight the advantages in the *P-S-N* curves estimation, see Figs. 7 and 8. Moreover, consistent with the results in Tables 5 and 6, as well as our expectations, the *P-S-N* curves estimating results of the hierarchical Bayesian model are more conservative than the MLE results. This is because hierarchical Bayesian model incorporates parameters and hyperparameter estimation error into the analysis and then meets more reasonable results, even

Table 2
Estimation results of hierarchical Bayesian model in Example 1 Case 1.

Node	Posterior			Posterior percentiles		
	Mean	SD	MC error	2.5%	Median	97.5%
α	4.983	0.01263	9.77E-5	4.958	4.983	5.008
β	−3.387	0.1102	9.123E-4	−3.603	−3.387	−3.171
σ	0.09617	0.009221	6.866E-5	0.08028	0.09547	0.1162
μ_α	8.369	320.2	2.125	−697.1	4.983	769.0
σ_α	1098.0	28920.0	206.8	0.04523	11.2	3941.0
μ_β	−4.047	294.4	2.069	−661.8	−3.381	624.2
σ_β	5006.0	605100.0	4289.0	0.0418	7.023	3199.0

Table 3
Estimation results of hierarchical Bayesian model Example 2 Case 1.

Node	Posterior			Posterior percentiles		
	Mean	SD	MC error	2.5%	Median	97.5%
α	5.582	0.02203	1.701E-4	5.538	5.582	5.626
β	−7.925	0.5109	0.006555	−8.917	−7.93	−6.916
σ	0.1602	0.01636	1.542E-4	0.1319	0.1588	0.1963
σ_α	1212.0	59100.0	417.4	0.04438	10.13	4016.0
μ_α	3.357	302.9	2.074	−690.1	5.584	670.6
σ_β	1546.0	76900.0	547.6	0.0437	9.57	3712.0
μ_β	−9.968	310.6	1.848	−733.3	−7.952	687.0

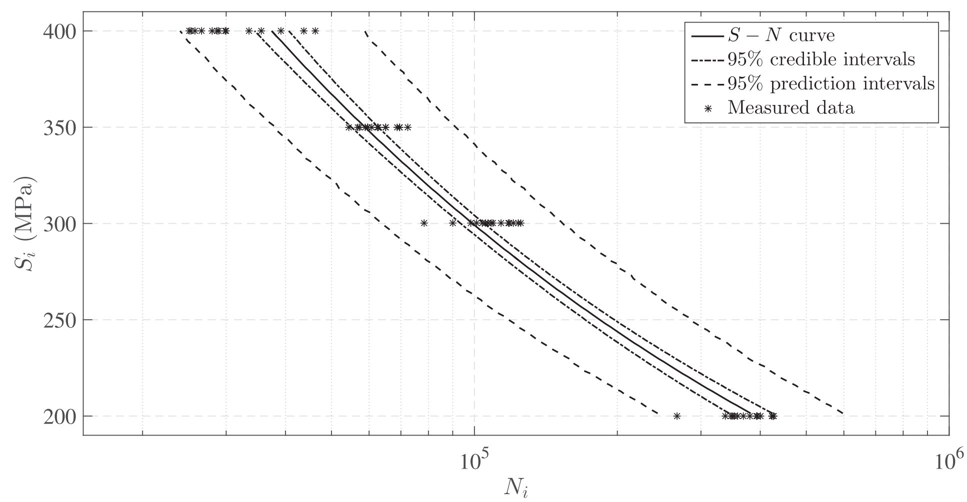


Fig. 3. The *S-N* curve fitting and *P-S-N* curves estimation, with 97.5% survival probability or 95% prediction intervals, in Example 1.

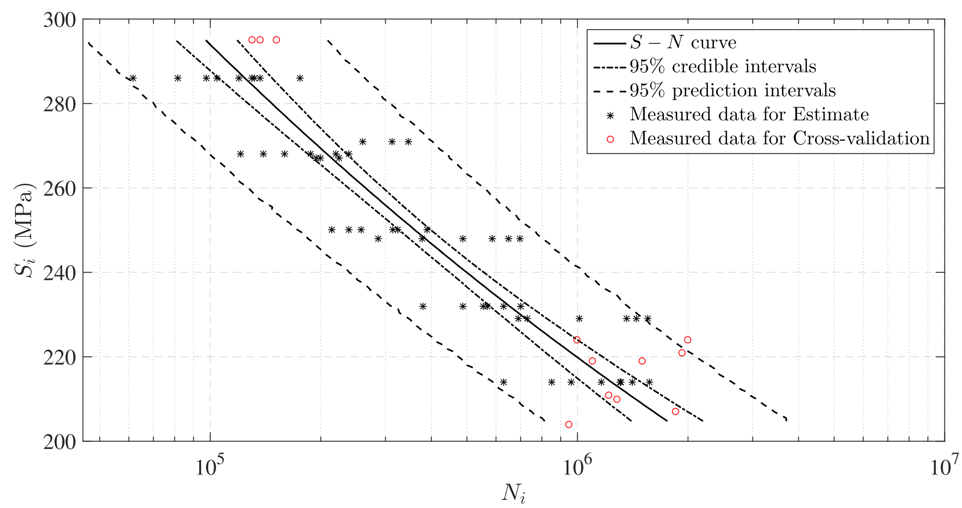


Fig. 4. The *S-N* curve fitting and *P-S-N* curves estimation, with 97.5% survival probability or 95% prediction intervals, in Example 2.

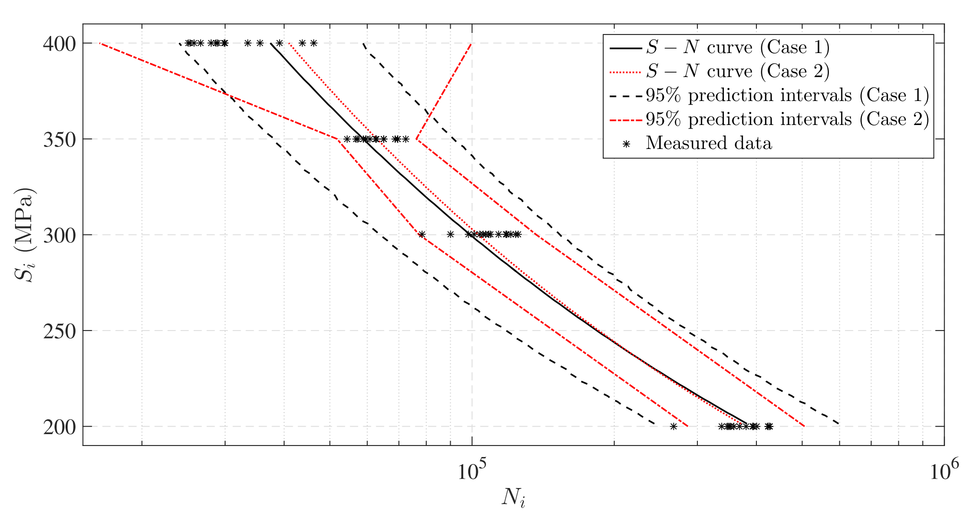


Fig. 5. A comparison of the hierarchical Bayesian model Case 1 and Case 2 in Example 1.

for a common variance setting. Its performance for the variances is that $\sigma_{y_i^{\text{pred}}} > \sigma_i > \sigma_i^{\text{MLE}}$ always happened, as shown in Tables 5 and 6. Additionally, the calculation of the missing data contained in the hierarchical Bayesian model is another influencing factor for its conservative results. To summarize, the hierarchical Bayesian model shows a significant advantage for the *P-S-N* curves estima-

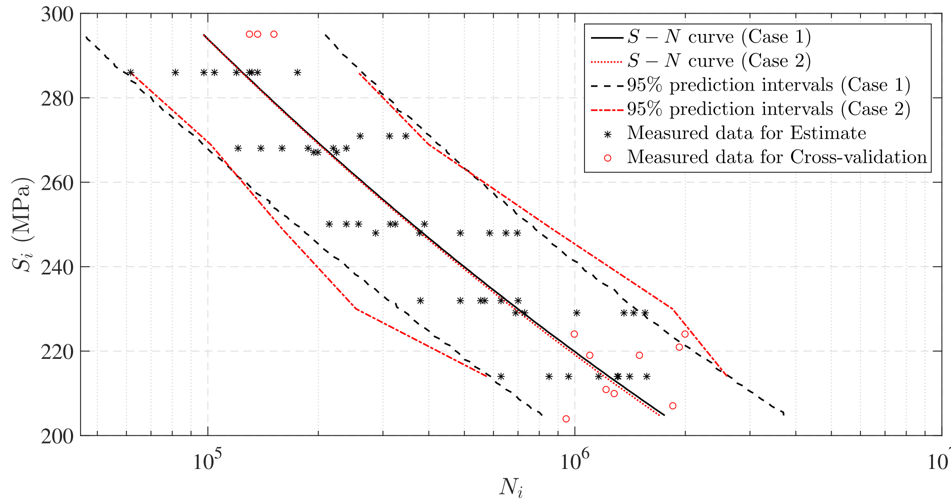


Fig. 6. A comparison of the hierarchical Bayesian model Case 1 and Case 2 in Example 2.

Table 4
The DIC values of examples.

Examples	Modle	$\overline{D}$	$\hat{D}$	p_D	DIC
Example 1	Case 1	−105.665	−108.670	3.005	−102.660
	Case 2	−145.930	−152.139	6.209	−139.721
Example 2	Case 1	−41.812	−44.867	3.055	−38.757
	Case 2	−40.804	−48.004	7.200	−33.603

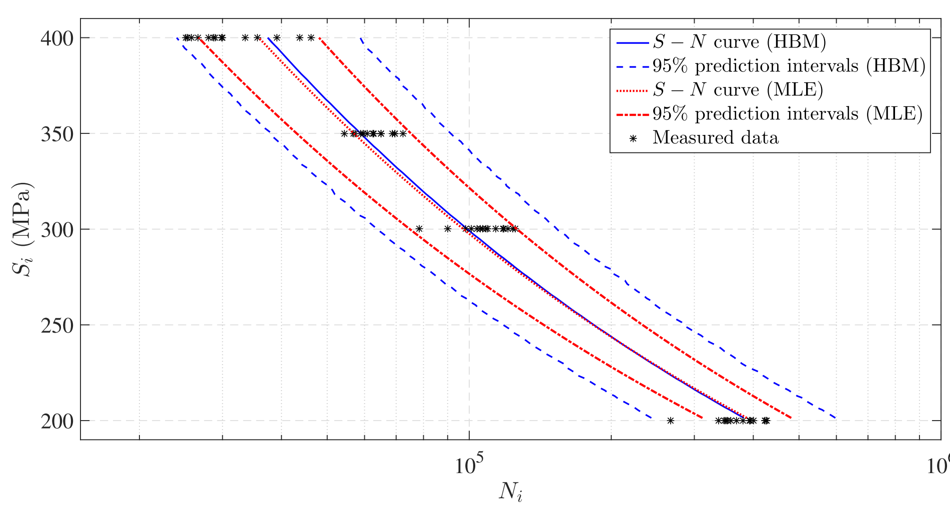


Fig. 7. A comparison of the hierarchical Bayesian model (Case 1) and MLE to estimate *S-N* curve and *P-S-N* curves in Example 1.

tion, while the likelihood-based methods are considerably faster in computational efficiency than the hierarchical Bayesian model with the MCMC algorithm involved.

Due to the limitations of the regression models, both the MLE and the hierarchical Bayesian model do not have very good performance at the highest stress levels for the *S-N* curve fitting in Example 1. In fact, the scatter is less at high-stress amplitudes, while larger at low-stress amplitudes [10]. Considering that the logarithmic coordinate display is non-linear, the display magnifies the error at high-stress levels (low lifetime) is understandable (Fig. 5). Furthermore, it seems that there is a knee point between stress levels 350 and 400 in Example 1. Thus, it is hard to obtain a better fitting result without a more complex model than Basquins relation, Kohout and Vechet function [48], for example. Since model selection is not the main focus of this paper, it will not be presented a detailed analysis here.

The three parameters model with structural noninformative prior also has been attempted to estimate the fatigue curves, as an extra trial. The prior of the third parameter, γ, is set to $\gamma \sim \mathcal{N}(\mu_\gamma, \tau_\gamma)$, and the prior of hyperparameters are set as conjugate priors, that is $\mu_\gamma \sim \mathcal{N}(0, 10^6)$, $\tau_\gamma \sim \mathcal{G}(0.001, 0.001)$. Because there is a logarithm scale of the parameter γ, the three parameters model is difficult to set a reasonable initial value for avoiding numerical overflow. This problem has been solved by setting $\tau_\gamma \sim \mathcal{G}(1, 1)$, limiting the variance of γ to a smaller range. Since the estimate of γ is very close to zero, the result of the three parameters model is close to the two parameters model. The smooth density curves, good BGR diagnostics result are obtained by the MCMC convergence monitors of the three parameters model, but a high autocorrelation. High autocorrelation leads to smaller effective sample size, it means that a subjective prior of γ with informative prior must be given.

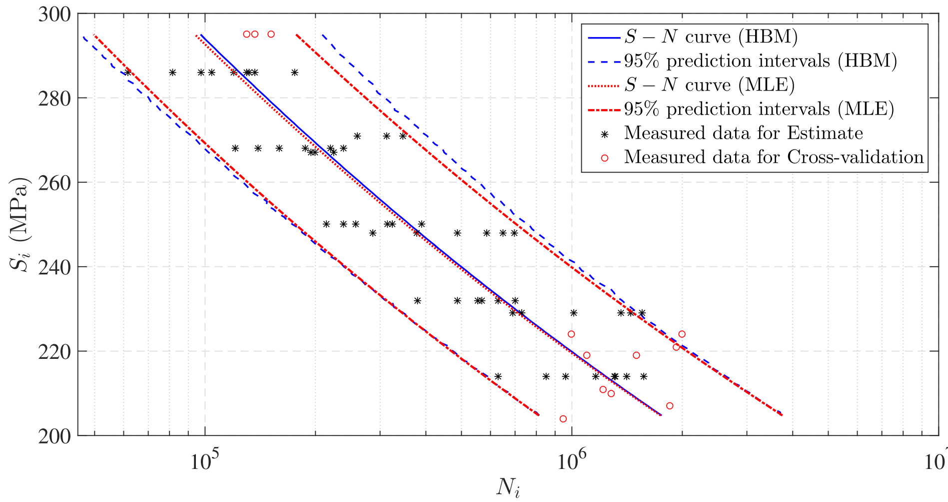


Fig. 8. A comparison of the hierarchical Bayesian model (Case 1) and MLE to estimate *S-N* curve and *P-S-N* curves in Example 2.

Table 5
The comparison of the standard deviations in Example 1.

Cases	SD	$\log_{10}N_1$	$\log_{10}N_2$	$\log_{10}N_3$	$\log_{10}N_4$
Case 1	σ		0.09617		
	$\sigma_{y_i^{pred}}$	0.0996	0.09588	0.09891	0.09777
Case 2	σ_i	0.05881	0.06053	0.04056	0.1939
	$\sigma_{y_i^{pred}}$	0.06243	0.06239	0.04202	0.1983
MLE	σ_i^{MLE}	0.05395	0.05485	0.03676	0.08711

Table 6
The comparison of the standard deviations in Example 2.

Cases	SD	$\log_{10}N_1$	$\log_{10}N_2$	$\log_{10}N_3$	$\log_{10}N_4$	$\log_{10}N_5$
Case 1	σ			0.1602		
	$\sigma_{y_i^{pred}}$	0.1664	0.1622	0.1607	0.1629	0.1643
Case 2	σ_i	0.1519	0.2106	0.1794	0.1413	0.1472
	$\sigma_{y_i^{pred}}$	0.1635	0.2188	0.1846	0.1483	0.1557
MLE	σ_i^{MLE}	0.1323	0.1968	0.1731	0.1339	0.1352

6. Conclusions

The hierarchical Bayesian model for estimating *S-N* curve have been presented in order that the test time and the number of specimens can be minimized, getting benefit from the hierarchical structure robustifies the usual Bayesian analysis. Following this, a *P-S-N* curves estimation method is proposed by using the predictive distributions, with the same hierarchical structure. The following conclusions can be drawn by numerical examples:

(1) Compared with the MLE, the hierarchical Bayesian model obtains more conservative results for *P-S-N* curves estimation, because the parameters and hyperparameter estimation errors are incorporated into the analysis, as well as the uncertainty effects of missing data. As a result, the hierarchical Bayesian model meets a safer design curve from small censored data sets.
(2) Benefit from the advantage of the hierarchical Bayesian model that can incorporate actual subjective prior information at the second stage, the physical models, such as random fatigue limit model, can be incorporated into a hierarchical Bayesian model conveniently for reducing the uncertainties. Besides, it allows the use of both structural prior information and subjective prior information simultaneously in fatigue data analysis. In addition, the physical model also can be set as a likelihood function for updating.
(3) Missing data can be easily handled in the hierarchical Bayesian model, it is helpful when the data is insufficient. Because of the more missing data means more uncertainties, and the results will achieve in larger prediction intervals.
(4) MCMC methods are broadly applicable, but require care in parametrization and convergence diagnosis.

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Appendix A. Figures of MCMC convergence monitoring

See Fig. A.1.

Appendix B. The original data

See Tables B.1 and B.2.

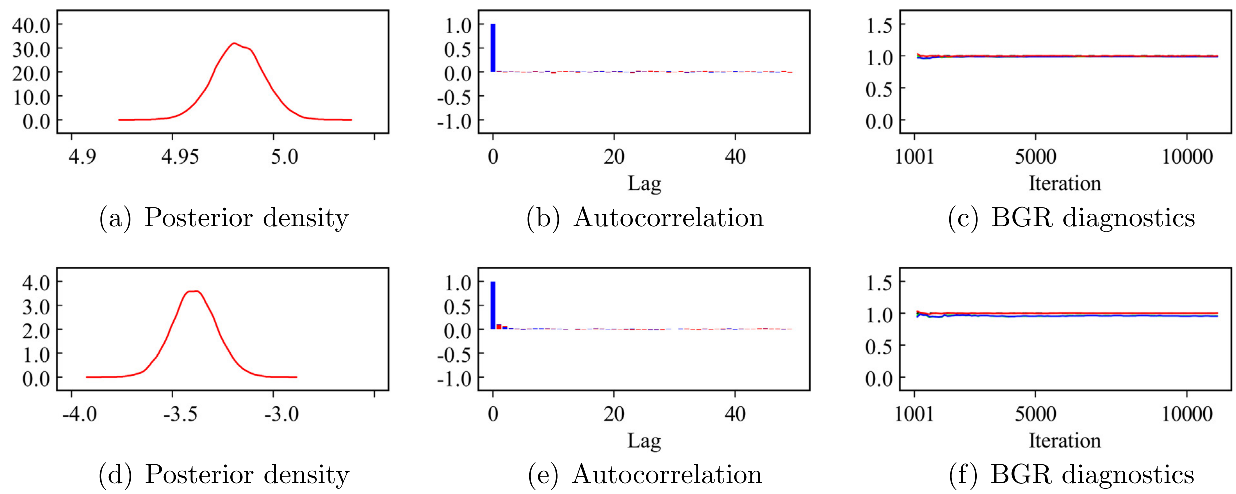

Fig. A.1. Results of MCMC convergence monitoring in Example 1, where sub-graph (a), (b), (c) for α and (d), (e), (f) for β.

Table B.1
Fatigue life test data of aluminum alloy 2524-T3 [13].

S_i (MPa)	$\log_{10} N_i$
200	5.603, 5.544, 5.528, 5.630, 5.594, 5.540, 5.581, 5.548, 5.426, 5.567, 5.554, 5.627, 5.630, 5.596, 5.626
300	5.028, 5.074, 5.016, 4.894, 4.993, 5.071, 5.024, 5.035, 4.954, 5.039, 5.098, 5.057, 5.092, 5.082, 5.005
350	4.784, 4.842, 4.776, 4.813, 4.813, 4.860, 4.798, 4.776, 4.758, 4.770, 4.755, 4.837, 4.736, 4.842, 4.796
400	4.477, 4.400, 4.426, 4.462, 4.592, 4.411, 4.447, 4.402, 4.665, 4.475, 4.458, 4.551, 4.525, 4.641

Table B.2
List of experimental durability data for the S420MC steel [46].

S_i (MPa)	N_i	S_i (MPa)	N_i	S_i (MPa)	N_i
204	946200	229	1447200	268	120800
207	1851500	232	488400	268	139800
210	1281700	232	380500	268	159100
211	1215000	232	567000	268	187100
214	628100	232	701800	268	219600
214	1307600	232	553000	268	238600
214	1316000	232	630000	271	259500
214	1410600	248	286700	271	313000
214	851900	248	376900	271	346100
214	1566600	248	488300	286	61600
214	959900	248	650100	286	119400
214	1159400	248	585900	286	81600
219	1095800	248	698500	286	132000
219	1499200	250	313700	286	130000
221	1926800	250	256900	286	104300
224	1999500	250	238800	286	97400
224	997600	250	323500	286	175600
229	690600	250	213700	286	136500
229	730500	250	389000	295	136800
229	1009600	267	199400	295	129900
229	1555800	267	194000	295	151400
229	1358000	267	224800		

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