

# General formulation of equivalent moment factor for elastic lateral torsional buckling of slender rectangular sections and I-sections

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## ABSTRACT

In the past decades, great progress has been made in analyzing lateral torsional buckling of slender beams. The phenomena has been accurately described by differential equations, closed form solutions are available for specific cases and the solution for any load and any boundary condition can be obtained by finite element analysis. Timber and steel design standards provide a procedure based on equivalent moment factors. With this procedure, beams can be designed straightforwardly. However, modern designers continue to push the envelope and more irregular load patterns are found, on which the design standards do not provide solutions. Consequently, designers are forced to determine the equivalent moment factors based on case-specific literature and/or conservative assumptions. Unfortunately, this makes many challenging modern designs uneconomical. Furthermore, significant inconsistencies between the different design procedures are found. For that purpose, this paper proposes a solution in the form of a general formulation to determine equivalent moment factors for any loading on a single-span beam for both free and restrained lateral bending and/or warping at the supports, for both I-sections and rectangular slender sections loaded in the shear center. It is shown that the obtained moment factors are accurate and in good agreement with design standards and literature, and a wide range of irregular load patterns is considered.

## 1. Introduction

The occurrence of elastic lateral torsional buckling (abbreviated as LTB) means a loss of structural stability and therefore, it is considered as an ultimate limit state failure mode that has to be addressed during the design of especially slender long-span laterally unsupported beams. LTB is the rotation of transversely loaded beams out of plane as a result of buckling of the compressive part of the cross-section due to bending. According to Lindner [1], the load level for which lateral torsional buckling takes place, called the critical load, depends on beam geometry, boundary conditions (rotational and translational restraints at the supports), distribution of the loading over the length of the beam, location of the loading within the cross-section, material properties and, as discussed by Pi and Trahair [2], pre-buckling deformations. This paper focuses on the influence of the load distribution and boundary conditions on the elastic critical load via the so-called *equivalent moment factor*, also called the moment gradient correction factor, for symmetrical beams loaded at the shear centre by uniform loads, point loads and bending moments at the supports (e.g. as shown in Fig. 4) and supported by fork supports, with the possibility to prevent warping as an additional feature. The equivalent moment factor is used to transfer

any loading situation into an equivalent simply supported beam that is loaded by a uniform bending moment, for which the critical load can be easily found. In this way, the critical theoretical bending moment is calculated based on the equivalent moment factor. However, in order to take account of material-specific phenomena like imperfections and residual stresses, this theoretical critical bending moment is subsequently reduced by a material-specific reduction factor. In this study, merely I-sections and rectangular slender sections, which are often made from steel and timber respectively, are considered.

During the last few decades, research on lateral torsional buckling focused on finite element formulations of advanced problems, due to the increasing complexity of solving the governing differential equations, as described by Timoshenko and Gere [3], for non-standard loading cases. As a result of the increasing application of computer programs in the design of structures, a clear understanding of buckling phenomena is becoming more relevant to make optimal use of the structural capacity. In this study, an analytic approach based on the principle of conservation of energy is followed, leading to analytic expressions which can be used by the designer to obtain more insight in the structural behaviour regarding lateral torsional buckling. Although analytic expressions might not be as accurate as numerical analyses, the

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simplicity of these expressions offers the possibility to quickly converge to the optimal design parameters of the considered structure, which is especially relevant for early design stages. However, the current design codes (such as EN 1993-1-1 [4], EN 1995-1-1 [5], AS-4100 [6] and ANSI/AISC 360-16 [7]) only offer solutions for basic load cases (e.g. mid-span point loads, uniformly distributed loads, clamped beams) and simplified general formulations that might lead to unnoticed significant errors or over-conservative results, hence creating the need for a general accurate analytic approach, considering all load combinations within a predefined framework, to evaluate lateral torsional buckling. At the moment, engineers are often referred to case-specific literature and are making use of rules of thumb (e.g. taking the effective buckling length equal to the distance between zero-moment-points, also known as points of contraflexure). This paper describes a general closed-form analytic approach to obtain the equivalent moment factor, which is one of the parameters that influences the total buckling behaviour, without over-simplifying the problem. This closed-form expression can be directly used by the designer or implemented in structural analysis software to incorporate design check procedures for lateral torsional buckling.

First, the current design codes are summarized to show the design procedures, followed by a review of the relevant literature. Significant inconsistencies between the different design codes and approaches in literature are found. Subsequently, the theoretical framework and the general theory are presented and applied to some specific cases. Finally, the general formulation is compared with literature, showing good agreement, and the limitations of the approach are discussed.

## 2. Design code procedures

The critical moment regarding lateral torsional buckling for double symmetric beams loaded in the shear centre is found by Timoshenko and Gere [3]:

$$M_{cr} = C_1 \frac{\pi^2 EI_z}{(kL)^2} \sqrt{\left(\frac{k}{k_w}\right)^2 \frac{J_w}{I_z} + \frac{(kL)^2 GI_t}{\pi^2 EI_z}} \quad (1)$$

in which coefficient  $C_1$  is the equivalent moment factor accounting for the non-uniform moment distribution. The influence of the support conditions on the length over which the beam buckles are considered by the lateral bending coefficient  $k$  and the warping coefficient  $k_w$ . For free lateral bending and/or warping at the supports, the values are equal to 1 and for prevented lateral bending and/or warping at the supports, the values become 0.5 (equivalent to a column that is clamped on both sides). It is noted that in practice, complete prevention of warping at the supports is unrealistic and therefore, prevented warping is merely interpreted as a theoretical limit case. The fork support as is shown in Fig. 2b offers both free lateral bending and warping. For I-sections, the moment resistance is a combination of warping and uniform (Saint-Venant) torsion. For rectangular slender sections, the effects of warping tend to be negligible, as discussed by Trahair [8] and Chajes [9], which reduces Eq. (1) to

$$M_{cr} = C_1 \frac{\pi}{kL} \sqrt{EI_z GI_t} \quad (2)$$

Eq. (1) is used in steel standards and Eq. (2) is used in timber standards, as most of the applied timber sections are rectangular and slender. In fact, lateral torsional buckling of a slender rectangular section follows the same expressions as an I-section with neglected warping. Hence, this study treats rectangular slender sections as if they are I-sections to generalize the approach. In this section, the calculation of the critical moment for lateral torsional buckling according to several design standards is elaborated.

**Table 1**

Equivalent moment factors according to EC3 [4].

Load case	$C_1$
Uniform moment ( $\psi = 1$ )	1.00
Linear moment from M to 0 ( $\psi = 0$ )	1.75
Linear moment from M to -M ( $\psi = -1$ )	2.30
Mid-span point load	1.35
Point loads on 0.25L and 0.75L	1.04
Uniformly distributed load	1.13

### 2.1. Eurocode 3 for steel

Lateral torsional buckling of double symmetric steel beams is considered in Eurocode EN 1993-1-1 [4] for steel structures as an ultimate limit state. For a beam loaded in the shear centre, EC3 gives for the critical bending moment

$$M_{cr} = C_1 \frac{\pi}{L_b} \sqrt{EI_z GI_t} \sqrt{1 + \frac{\pi^2 EI_w}{L_b^2 GI_t}} \quad (3)$$

Eqs. (1) and (3) are identical when  $k$  and  $k_w$  have the same value and the buckling length  $L_b$  is set to  $kL$ . Using the approach of EC3, it is not possible to have different values for the lateral bending and warping coefficients. The equivalent moment factor  $C_1$  is prescribed for basic load cases based on the work of Gardner et al. [10] and Trahair [8], as shown by Table 1. For a simply supported beam loaded at the supports by a moment  $M$  and  $\psi M$ , with  $-1 \leq \psi \leq 1$ , EC3 gives an analytic expression:  $C_1 = 1.75 - 1.05\psi + 0.3\psi^2 \leq 2.3$ . Furthermore, EC3 gives design diagrams for obtaining  $C_1$  for the loading cases with support moments combined with uniformly distributed loads or mid-span point loads, based on the work of Bijlaard and Steenbergen [11].

### 2.2. Eurocode 5 for timber

The Eurocode EN 1995-1-1 [5] for timber structures provides guidelines for transversely loaded rectangular slender sections, for which warping is neglected. The critical moment is given by

$$M_{cr} = \frac{\pi}{L_{eff}} \sqrt{EI_z GI_t} \quad (4)$$

in which  $L_{eff}$  is the effective buckling length. For a lateral bending and warping free beam, such that  $k = k_w = 1$ , the effective length is equal to  $L/C_1$  and hence, Eq. (4) is equal to Eq. (2). The current version of EC5 only gives the effective length for 3 cases: for uniform moments 1.00, for uniformly distributed loads 0.90 and for mid-span point loads 0.80. For other loadcases, the designer is referred to literature or other design standards.

### 2.3. Australian steel standards

The Australian Steel Standards AS-4100 [6] make use of Eq. (1). Instead of equivalent moment factors for specific load-cases, the AS-4100 provides a general expression for beams that are laterally bending and warping free:

$$C_1 = \frac{1.7M_{max}}{\sqrt{M_a^2 + M_b^2 + M_c^2}} \quad (5)$$

where  $M_{max}$  is the maximum absolute moment within the span and  $M_a$ ,  $M_b$  and  $M_c$  are the absolute values of the moments at respectively  $L/4$ ,  $L/2$  and  $3L/4$  of the span.

### 2.4. American standards

Both the American Steel Standard (ANSI/AISC 360-16 [7]) and the Timber Standard (AFPA-TR14 [12]) give the same closed-form

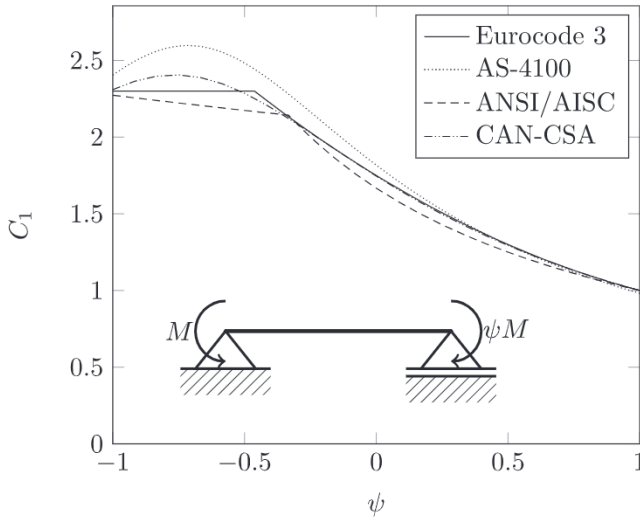


Fig. 1.  $C_1$  as function of  $\beta$  for the different standards.

expression to calculate the equivalent moment factor for a given moment distribution, based on the work of Kirby and Nethercot [13].

$$C_1 = \frac{12.5M_{max}}{2.5M_{max} + 3M_a + 4M_b + 3M_c} \quad (6)$$

### 2.5. Canadian standards

The Canadian Steel Standard CAN-CSA S16-14 [14] provides the following closed-form expression for  $C_1$ :

$$C_1 = \frac{4M_{max}}{\sqrt{M_{max}^2 + 4M_a^2 + 7M_b^2 + 4M_c^2}} \leq 2.5 \quad (7)$$

Fig. 1 compares the expressions of the considered design standards for a linear moment gradient on a simply supported beam. The standards are in good correspondence with each other, although differences can be significant. According to Sahraei et al. [15], the Australian standard delivers, with respect to finite element analysis, the most accurate solution for basic load cases.

### 3. Literature review

Lateral torsional buckling of beams can be studied in two ways, as mentioned by Pi et al. [16]. Firstly, by considering the nonlinear differential equations, as posed by Timoshenko and Gere [3], which can be solved in closed form for a few simplistic cases. For more general cases finite integrals, series solution, finite differences, numerical integration and finite element methods can be used. Secondly, an energy-conservation based approach can be followed: the work done by the load during buckling must be equal to the increase in strain energy. Using approximate deformation shapes, the buckling load can be determined. The accuracy of energy-based methods is highly depending on the applied deformation shape.

Using an energy-based approach, Chajes [9] derived expressions for lateral-torsional buckling of simply or warping restraining supported I-sections for two load-cases: uniform bending and mid-span point loads. The books of Kirby and Nethercot [13] and Trahair [8] comprehend a more elaborate collection of load cases based on both energy approaches and finite element analyses. Trahair described amongst others simply supported and restrained beams loaded by one point load at arbitrary location, beams loaded by mid-span point loads combined with support bending moments and beams loaded by a uniformly distributed load combined with support bending moments. Although this set of load-cases is quite elaborate, only basic load-cases are considered

Table 2

$S_1$  and  $S_2$  values from Lim et al. [19].

$k$	$k_w$	$S_1$	$S_2$
1.0	1.0	1.00	0.16
0.5	0.5	1.00	0.18
1.0	0.5	0.80	0.10

and there is no general expression fulfilling all cases. This study aims to find an expression that is approximately valid for all load-cases, without over-simplifying the problem.

During the years, lots of experimental data have been obtained regarding the equivalent moment factor (for example by Xiao et al. [17] and Buraw et al. [18]). However, besides verification of theories, the use of experimental data is rather limited, since information is only obtained for specific cases. Both [17,18] show that the experimental and theoretical values for the critical buckling moments are close to each other for the considered cases and hence, this study will not further consider experimentally determined equivalent moment factors.

Using both the Bubnov-Galerkin method and the finite element method, Lim et al. [19] found an expression to capture the equivalent moment factor for linear moment gradients, as defined in Fig. 1, for elastic lateral torsional buckling of I-sections:

$$C_{1,Lim} = \frac{2}{\sqrt{S_1 \sqrt{(1+\psi)^2 + S_2(1-\psi)^2}}} \quad (8)$$

in which  $S_1$  and  $S_2$  are coefficients depending on the support conditions, as given in Table 2 for different lateral bending and warping restraints  $k$  and  $k_w$  respectively.

Serna et al. [20] studied  $C_1$  more in depth with the aid of a finite difference approach. Design tables and graphs were determined for a wide variety of cases (linear moments and uniformly distributed load or a concentrated load together with one or two support moments), loaded in the shear centre. They found that coefficient  $C_1$  is slightly dependent on the length of span, in line with the observations of Nethercot and Rockey [21]. Based on this elaborate analysis, Serna et al. proposed an improved expression for  $k = k_w = 1$  which governs the studied load cases accurately and conservatively, without taking into account the length dependency:

$$C_{1,Serna} = \sqrt{\frac{35M_{max}^2}{M_{max}^2 + 9M_a^2 + 16M_b^2 + 9M_c^2}} \quad (9)$$

Also following the finite difference approach, Suryatmono and Ho [22] found equivalent moment factors for beams under uniform loading and one or two support moments. Furthermore, they showed that significant differences occur when using the general formulas from design standards, leading to over-conservative designs. Sahraei et al. [15] considered elastic lateral torsional buckling of rectangular slender wooden beams and presented a simplified expression to find the equivalent moment factor for a wide variety of load cases:

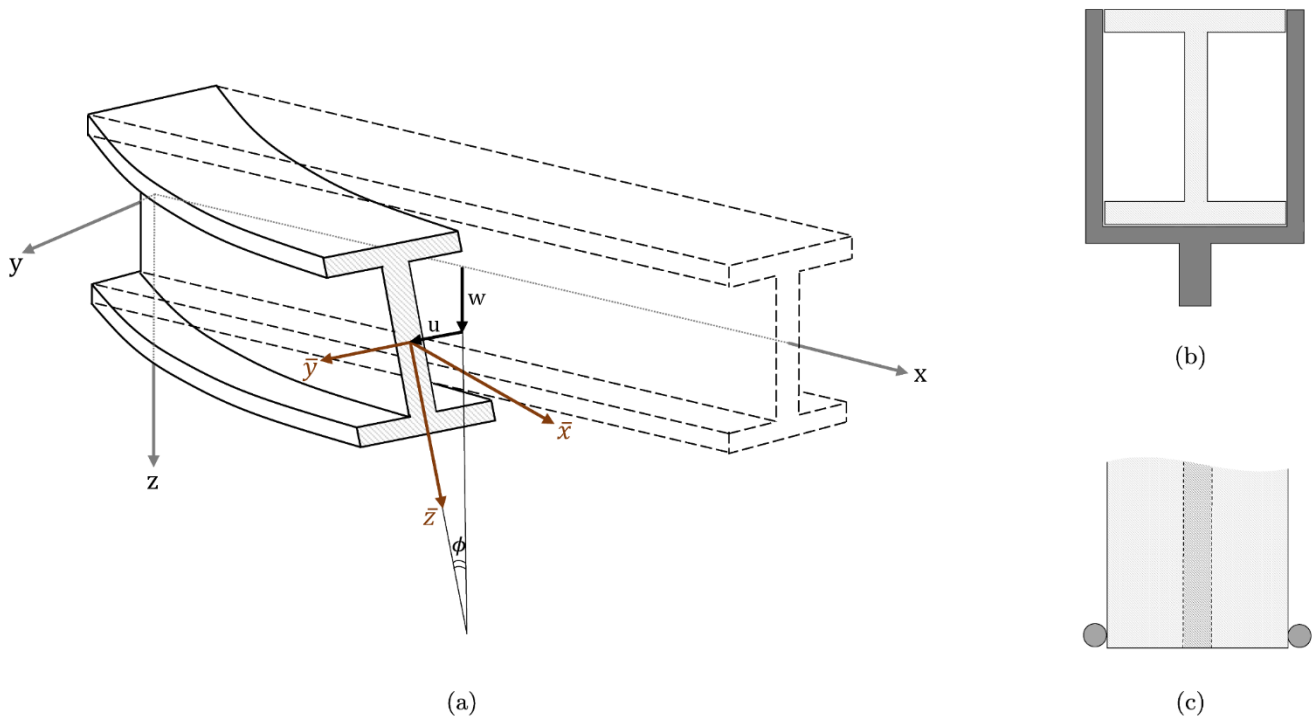
$$M_{cr,Sahraei} = C_r C_b C_L C_p \sqrt{EI_z G I_t} \quad (10)$$

in which

- $C_r$  accounts for partial twist restraint at beam-ends, meaning that the rotation out of plane is not fully fixed at the supports.
- $C_b$  is the equivalent moment factor accounting for non-uniform moment distributions, defined as  $C_1$  in the Eurocode 3 for steel [4].
- $C_L$  considers the influence of the load-location within the cross-section with reference to the shear centre.
- $C_p$  is a coefficient that accounts for pre-buckling deformations (the so-called second order effects).

In their study, Sahraei et al. derived detailed expressions for  $C_r$ ,  $C_L$  and  $C_p$ . For  $C_b$ , the equivalent moment factor, the expression of the





**Fig. 2.** (a) Definitions of the coordinate-systems in undeformed ( $x$ ,  $y$  and  $z$ ) and deformed configuration (post-buckling,  $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$ ) with displacements  $u$ ,  $w$  and  $\phi$ ; (b) Side view of fork support; (c) Top view of fork support.

Australian Standard (Eq. (5)) is adopted, as this expression compared best to finite element analyses of standard load cases. However, it is not known whether Eq. (5) performs sufficiently accurate for more irregular loading cases. Furthermore, the strength of a simplistic expression becomes questionable when more complex cases are studied, since these expressions are most likely over-simplifications of the physical problem and might not be accurate. The aim of this study is to find a more generally applicable closed-form expression for the equivalent moment factor, which subsequently can be implemented in the study of Sahraei et al. to further improve the accuracy of Eq. (10). Furthermore, this closed-form expression can be used in the design to consider irregular load patterns, without the need to perform an elaborate finite element analysis or study case-specific literature.

#### 4. Theoretical framework

The beam is considered in two configurations: the undeformed configuration ( $x$ ,  $y$  and  $z$ ) and the deformed configuration ( $\bar{x}$ ,  $\bar{y}$  and  $\bar{z}$ ) just after the occurrence of lateral torsional buckling with displacements  $u$ ,  $w$  and  $\phi$ . Both of the configurations are defined in Fig. 2a. For the purpose of this paper, the following key assumptions are made:

1. The beam is prismatic with an I-section. The theory also holds for slender rectangular sections, by implementing  $I_w \approx 0$  in Eq. (1). Because only double symmetric sections are studied, bending moments around the different axes are uncoupled.
2. The Euler-Bernoulli beam theory is applied, meaning that plane sections remain plane and shear deformations are not considered.
3. The loading acts transversely in the shear centre of the beam, not causing any additional torsional loading on the beam. No lateral loading is included.
4. Displacements are assumed to be small such that first order approximations of the deformed configuration are valid and pre-buckling deformations can be neglected.

5. Elastic material behaviour is assumed, meaning that no energy dissipation can take place.
6. The considered beam is supported on both sides by fork supports, as shown in Fig. 2b and c. The fork supports prevent lateral displacements  $u$  and twisting  $\phi$  and allow for lateral bending. In this study, the possibility to prevent warping ( $\phi = 0$ ) at the supports is also included. However, both of the supports should either have warping restrained or allowed. In the span, the beam is laterally unsupported.
7. The beam is not loaded in axial direction.

In the derivation of Section 5, an energy-based approach is followed: the work done by the loading must be equal to the change in strain energy. Prior to lateral torsional buckling, the beam deforms merely within the (vertical) plane of loading, causing transverse displacements  $w$ . Just after the occurrence of lateral torsional buckling, an out of plane rotation  $\phi$  is found. The beam is transversely loaded, following key assumption 3, and therefore, work can only be done within this considered plane. To this end, the transverse displacements during lateral torsional buckling  $w_{LTB}$ , which are directly related to the rotation  $\phi$  and merely take place as a result of this rotation, are required for this energy-based approach.

#### 5. Derivation of general formulation

The derivation of a general formulation for the equivalent moment factor  $C_1$  is based on the energy approach of the American Wood Council [12].  $C_1$  describes the ratio between the critical load of a beam loaded by a uniform bending moment (Fig. 3) and a beam loaded by arbitrary loading and hence, both must be considered. For the purpose of simplicity, the arbitrary loading is first simplified to a single point load and two end-moments (Fig. 4). Later on, arbitrary loading is considered as a superposition of multiple point loads and a general theory is formulated. In this way, any loading can be simulated by representing it as a collection of point loads.

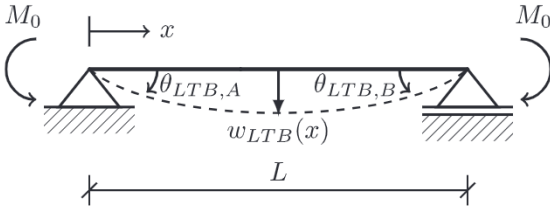


Fig. 3. Simply supported beam loaded by a uniform bending moment  $M_0$  with deformations due to LTB in plane of loading.

### 5.1. General considerations

In this derivation, an energy-based approach is followed. The energy prior to buckling is equaled with the energy just after buckling, following assumption 5. Since the total amount of energy in the system stays constant, the increase in energy

$$\Delta E = \Delta U + \Delta V = 0 \quad (11)$$

with  $U$  the strain energy and  $V$  the potential energy. For the considered I-section and following the key assumptions, strain energy can be the result of bending (around both y- and z-axes), torsion and warping. For a beam loaded in bending around a certain axis, the strain energy  $U_b$  taken by the beam is found by integrating the internal work over the complete volume.

$$U_b = \frac{1}{2} \int_V \sigma_b \varepsilon_b dV \quad (12)$$

Combining Eq. (12) with Hooke's law ( $\sigma_b = E\varepsilon_b$ ), the kinematic relation  $\varepsilon_b = \kappa_z z / EI$  and the moment of inertia  $I_y = \int_A z^2 dA$ , leads to

$$U_b = \frac{1}{2} \int_0^L \frac{M_b^2}{EI} dx \quad (13)$$

In the same manner the strain energy taken by warping  $U_w$  and torsion  $U_t$  are derived (see [23,9] for complete derivations).

$$U_w = \frac{1}{2} \int_0^L \frac{M_w^2}{EI_w} dx \quad (14)$$

$$U_t = \frac{1}{2} \int_0^L \frac{M_t^2}{GI_t} dx \quad (15)$$

The potential energy is obtained by the position of the loading with respect to a certain reference state. For the considered problem, the change in potential energy  $\Delta V$  is equal but opposite to the work done by the loading in the considered interval. Since the deformations are assumed to be small (assumption 4), the bending moments in the undeformed and deformed configuration can be related by  $M_{\bar{y}} \approx M_y$  and  $M_{\bar{z}} \approx \phi M_y$ .

### 5.2. Uniform moment over beam

A simply supported beam loaded by a uniform bending moment  $M_0$

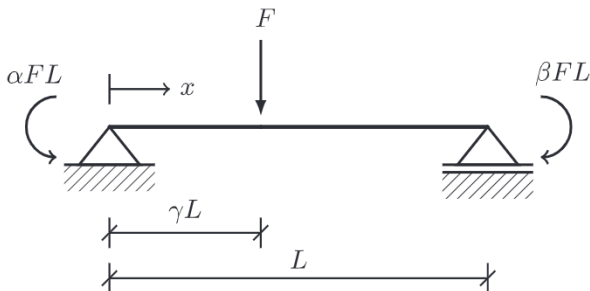


Fig. 4. Simply supported beam loaded by a point load and end-moments with the corresponding bending moment line  $M_y$ .

is shown in Fig. 3. Prior to lateral torsional buckling, only bending around the y-axis takes place.

$$U_I = U_{b,y} = \frac{1}{2} \int_0^L \frac{M_y^2}{EI_y} dx \quad (16)$$

Just after lateral torsional buckling, bending around both  $\bar{y}$  and  $\bar{z}$ -axes, warping and torsion are considered in the deformed configuration, leading to

$$U_{II} = U_{b,\bar{y}} + U_{b,\bar{z}} + U_{t,\bar{x}} + U_{\bar{w}} = \frac{1}{2} \int_0^L \left\{ \frac{M_{\bar{y}}^2}{EI_y} + \frac{M_{\bar{z}}^2}{EI_z} + \frac{M_{\bar{x}}^2}{GI_t} + \frac{M_{\bar{w}}^2}{EI_w} \right\} dx \quad (17)$$

Using the relations between the deformed and undeformed configurations and assuming small deformations do not influence warping and torsional behaviour, Eq. (17) can be elaborated to

$$U_{II} = \frac{1}{2} \int_0^L \left\{ \frac{M_y^2}{EI_y} + \frac{(\phi M_y)^2}{EI_z} + \frac{M_x^2}{GI_t} + \frac{M_w^2}{EI_w} \right\} dx \quad (18)$$

Next, the change in strain energy is found by  $\Delta U = U_{II} - U_I$ , giving with Eqs. (16) and (18)

$$\Delta U = \frac{1}{2} \int_0^L \left\{ \frac{(\phi M_y)^2}{EI_z} + \frac{M_x^2}{GI_t} + \frac{M_w^2}{EI_w} \right\} dx \quad (19)$$

The change in potential energy  $\Delta V$  is equal and opposite to the work done by the loading. For the considered beam, the only work is done by the end moments, giving

$$\Delta V = -(\theta_{LTB,A} + \theta_{LTB,B}) \cdot M_0 = -2\theta_{LTB,A} \cdot M_0 \quad (20)$$

in which  $\theta_{LTB,A}$  and  $\theta_{LTB,B}$  are defined in Fig. 3. The transverse displacement  $w_{LTB}$  due to only lateral torsional buckling is related to the lateral displacement  $u_{LTB}$  via  $w_{LTB} = \phi u_{LTB}$ . Furthermore,  $u_{LTB}$  is found with the moment-area theorem, whilst applying  $M_{\bar{z}} \approx \phi M_y$ , leading to

$$w_{LTB}(x_p) = x_p \theta_{LTB,A} - \int_0^{x_p} \phi \cdot \frac{\phi M_y}{EI_z} \cdot (x_p - x) dx \quad (21)$$

From Eq. (21),  $\theta_{LTB,A}$  can be solved by considering the transverse displacement  $w_{LTB}(x_p = L) = 0$ , giving

$$\theta_{LTB,A} = \int_0^L \frac{\phi^2 M_y}{EI_z} \frac{(L-x)}{L} dx \quad (22)$$

Following the same approach, an expression for  $\theta_{LTB,B}$  is found:

$$\theta_{LTB,B} = \int_0^L \frac{\phi^2 M_y}{EI_z} \frac{x}{L} dx \quad (23)$$

For a uniform bending moment  $M_y = M_0$  and assuming a buckling shape  $\phi$  that is symmetric around  $x = L/2$ , Eq. (22) further reduces to

$$\theta_{LTB,A} = \frac{M_0}{2EI_z} \int_0^L \phi^2 dx \quad (24)$$

Substitution of  $\theta_{LTB,A}$  in Eq. (20) results in

$$\begin{aligned} & -\alpha FL \quad -\beta FL \\ & \quad \quad \quad + \\ & \quad \quad \quad FL(\gamma(1-\gamma) - \alpha(1-\gamma) - \beta\gamma) \end{aligned}$$

$$\Delta V = -\frac{M_0^2}{EI_z} \int_0^L \phi^2 dx \quad (25)$$

Inserting Eqs. (19) and (25) in Eq. (11) leads to

$$\Delta E = \frac{1}{2} \int_0^L \left\{ -\frac{(\phi M_0)^2}{EI_z} + \frac{M_x^2}{GI_t} + \frac{M_w^2}{EI_w} \right\} dx = 0 \quad (26)$$

### 5.3. Point load and support-moments on beam

A simply supported beam loaded by a point load  $F$  at  $\gamma L$  (with  $0 < \gamma < 1$ ) and two end moments  $M_A = \alpha FL$  and  $M_B = \beta FL$  is shown in Fig. 4 together with the corresponding bending moment line. Since the point load acts in the same plane as the support-moments, the change in strain energy  $\Delta U$  is found by Eq. (19). Furthermore, the change in potential energy  $\Delta V$  is found in a similar way as Eq. (20), adding the change in potential of the point load:

$$\Delta V = -F w_{LTB}(\gamma L) - \theta_{LTB,A} M_A - \theta_{LTB,B} M_B \quad (27)$$

in which  $w_{LTB}$ ,  $\theta_{LTB,A}$  and  $\theta_{LTB,B}$  are given by Eqs. (21)–(23) respectively. The total energy change in the system is now found by

$$\frac{1}{2} \int_0^L \left\{ \frac{(\phi M_y)^2}{EI_z} + \frac{M_x^2}{GI_t} + \frac{M_w^2}{EI_w} \right\} dx + \Delta V = 0 \quad (28)$$

Considering the constitutive relations  $M_x = GI_t(d\phi/dx)$  and  $M_w = -EI_w(d^2\phi/dx^2)$ , and applying the same buckling shape  $\phi(x)$  for the uniform loading and the considered case, the torsion and warping terms in Eqs. (26) and (28) work out to be equal and hence, the other terms in these expressions should be equal as well. The accuracy of this approach depends on the similarity of the buckling shapes of the considered and uniform load cases, as will be discussed later on. Combining Eqs. (26) and (28) gives

$$\frac{1}{2} \int_0^L \left\{ \frac{(\phi M_y)^2}{EI_z} + \frac{(\phi M_{0,eq})^2}{EI_z} \right\} dx - F w_{LTB}(\gamma L) - \theta_{LTB,A} M_A - \theta_{LTB,B} M_B = 0 \quad (29)$$

which can be solved to obtain the equivalent uniform moment  $M_{0,eq}$ , once a certain buckling shape  $\phi(x)$  is specified. The equivalent moment factor  $C_1$  is now found by

$$C_1 = \frac{M_{max}}{M_{0,eq}} \quad (30)$$

### 5.4. Arbitrary point loading on a beam

In this study, arbitrary loading is defined as a random collection of  $N$  point loads  $\delta_i F$ , in which  $F$  is the unit load and  $\delta_i$  the collection of multipliers with  $i = 1..N$ , on a beam with support-moments  $\alpha FL$  and  $\beta FL$  on both sides. The loads are applied on locations  $\gamma_i$ . Intermediate moment loads are not considered but can be implemented by the reader by considering the loss of potential energy by this moment load during buckling in the same manner as the support-moments. Taking multiple point loads into account, Eq. (29) changes to

$$\frac{1}{2} \int_0^L \left\{ \frac{(\phi M_y)^2}{EI_z} + \frac{(\phi M_{0,eq})^2}{EI_z} \right\} dx - \sum_{i=1}^N \delta_i F w_{LTB}(\gamma_i L) - \theta_{LTB,A} M_A - \theta_{LTB,B} M_B = 0 \quad (31)$$

and the moment  $M_y$  is found by

$$M_y(x) = -\alpha F(L-x) - \beta Fx + \sum_{i=1}^N \begin{cases} \delta_i Fx(1-\gamma_i) & 0 \leq x \leq \gamma_i L \\ \delta_i F\gamma_i(L-x) & \gamma_i L \leq x \leq L \end{cases} \quad (32)$$

Different sizes of point loads can be included in two ways: either by the multipliers  $\delta_i$  or by manipulating the location collection  $\gamma_i$  such that multiple unit loads  $F$  on the same location represent a certain relative size. In the remainder of this paper, the latter approach is used in order to reduce the number of different input parameters. By implementing

Eqs. (31) and (32) in Eq. (30), a general expression has been derived to determine the equivalent moment factor  $C_1$  once a certain buckling shape  $\phi(x)$  is assumed. For a fork support that allows for warping ( $k = k_w = 1$ ), many authors (e.g. [3,8,9]) assume a sinusoidal buckling shape according to

$$\phi(x) = a \sin \frac{\pi x}{L} \quad (33)$$

with  $a$  being an unknown scaling variable. Eq. (33) fulfills the boundary conditions: twisting is prevented on both sides, so  $\phi(0) = \phi(L) = 0$ , and warping is allowed meaning  $M_w(0) = M_w(L) = 0$  and hence  $\phi''(0) = \phi''(L) = 0$ . Substitution of the assumed buckling shape  $\phi(x)$  in Eq. (31), solving for  $M_{0,eq}$  using the moment distribution as defined in Eq. (32) and inserting the obtained  $M_{0,eq}$  in Eq. (30) results in a general expression for the equivalent moment factor  $C_1$ :

$$C_1 = \frac{M_{max}}{FL \sqrt{D_1(\alpha^2 + \beta^2) + D_2\alpha\beta + D_3\alpha + D_4\beta + D_5}} \quad (34)$$

In which coefficients  $D_1$  to  $D_5$ , given in Appendix A, are fully defined by the collection of locations of point loads  $\gamma_i$ . Appendix A also gives an expression to calculate the maximum bending moment in the beam. For example, when a beam is loaded by point loads of 15 kN at  $L/2$  and 10 kN at  $3L/4$ ,  $\gamma_i = [0.5, 0.5, 0.5, 0.75, 0.75]$  with a unit point load  $F$  of 5 kN. In this manner, it is possible to include different sizes of point loads, although it is not possible to include point loads with different directions in this formulation. These can be included by elaborating Eq. (31) with an adjusted moment distribution. The coefficients  $D_1$  to  $D_5$  have a physical meaning.

- $D_1$  and  $D_2$  are constant values, respectively 0.2827 and 0.4347, taking account of the influence of the support-moments on the equivalent moment factor. These factors are constant as only the magnitude of the support-moments is susceptible to change.
- $D_5$  considers the influence of the point loads on  $C_1$  by the moment distribution and the work done by the point loads. For  $\alpha = \beta = 0$ , Eq. (34) reduces to  $M_{max}/(FL\sqrt{D_5})$ .
- $D_3$  and  $D_4$  are interaction factors for the point loads and the support-moments. The vertical displacements  $w_{LTB}(\gamma_i L)$  are influenced by the size of the support-moments and the support rotations  $\theta_{LTB}$  are influenced by the point loads, hence creating cross-terms.

Using the same formulations, it is possible to prevent warping at the supports while allowing for lateral bending ( $k = 1$  and  $k_w = 0.5$ ) by assuming a different buckling shape for both the uniform moment load case and the arbitrary loading

$$\phi(x) = a \left( 1 - \cos \frac{2\pi x}{L} \right) \quad (35)$$

which fulfills the boundary conditions:  $\phi(0) = \phi(L) = 0$  and  $\phi'(0) = \phi'(L) = 0$ , in which the latter condition describes prevented warping at the supports. As discussed by Trahair [8], the influence of warping-restraints at the supports on the critical moment depends on the torsional stiffness  $GI_t$ , the warping stiffness  $EI_w$  and the length  $L$  of the beam. However, these effects are completely neglected by assuming that Eq. (35) holds for both the uniform moment and the arbitrary loading. Restrained warping at the supports has a favourable influence on lateral torsional buckling, leading to larger  $C_1$  values. Tables 3 and 4 give coefficients  $D_3$ ,  $D_4$  and  $D_5$  for a single point load and multiple point loads respectively for both free and restrained warping at the supports. For prevented warping, coefficients  $D_1$  and  $D_2$  are constants: 0.2700 and 0.4600 respectively.

### 5.5. Uniformly distributed loading

Uniformly distributed loads  $q$  are handled in the same manner. The force-related term in the potential energy change  $\Delta V$  is replaced by an



**Table 3**Coefficients  $D_3$ ,  $D_4$  and  $D_5$  for 1 point load at  $\gamma L$  with and without warping allowed at the supports.

$\gamma$	0.1	0.2	0.25	0.33	0.4	0.5	0.6	0.67	0.75	0.8	0.9
$k_w = 1$											
$D_3$	-0.0562	-0.1087	-0.1312	-0.1592	-0.1735	-0.1757	-0.1581	-0.1369	-0.1069	-0.0864	-0.0434
$D_4$	-0.0434	-0.0864	-0.1069	-0.1369	-0.1581	-0.1757	-0.1735	-0.1592	-0.1312	-0.1087	-0.0562
$D_5$	0.0028	0.0106	0.0158	0.0241	0.0300	0.0335	0.0300	0.0241	0.0158	0.0106	0.0028
$k_w = 0.5$											
$D_3$	-0.0540	-0.1072	-0.1322	-0.1664	-0.1863	-0.1925	-0.1730	-0.1483	-0.1144	-0.0919	-0.0460
$D_4$	-0.0460	-0.0919	-0.1144	-0.1483	-0.1730	-0.1925	-0.1863	-0.1664	-0.1322	-0.1072	-0.0540
$D_5$	0.0027	0.0107	0.0163	0.0263	0.0340	0.0388	0.0340	0.0263	0.0163	0.0107	0.0027

integral over the complete length of the beam:

$$\sum_{i=1}^N \delta_i F_{wLTB}(\gamma_i L) = \int_0^L q \cdot w_{LTB}(\gamma L) d\gamma \quad (36)$$

and  $M_A = \alpha q L^2$  and  $M_B = \beta q L^2$  are implemented in Eq. (31), leading to

$$C_1 = \frac{M_{max}}{q L^2 \sqrt{D_1(\alpha^2 + \beta^2) + D_2\alpha\beta + D_3\alpha + D_4\beta + D_5}} \quad (37)$$

The coefficients  $D_1$  and  $D_2$  remain unchanged and coefficients  $D_3$ ,  $D_4$  and  $D_5$  are given in Table 4. For the combination of point loads and uniformly distributed loads, one should represent the UDL as a collection of equivalent equally-spaced point loads, such that the expressions in Appendix A together with Eq. (34) can be applied. This approach is further elaborated in Section 7.

## 6. Validation

In this section, the derived general formulations for point loads (Eq. (34)) and uniformly distributed loads (Eq. (37)) are compared with the available literature. In Table 5, equivalent moment factors are given for practical load-cases like a moment gradient from  $M$  to  $\psi M$ , uniformly distributed loads (UDL) and point loads (PL) for simply supported, single clamped (SC) and double clamped (DC) beams based on Eqs. (34) and (37), Eurocode 3 [4], the Australian Steel Standard AS-4100 [6], the American Steel Standard ANSI/AISC 360-17 [7], being equal to the American Timber Standard AFPA-TR14 [12], the collection of Trahair [8] and the general formulation from Serna et al. [20] as given by Eq. (9). Only supports with  $k = k_w = 1$  are considered in Table 5 since most design standards do not offer expressions to include support restraints. The average ratio between the found general formulations and literature lies between 1.04 and 1.09, with the lowest ratio for AS-4100, meaning that this study results in a non-conservative equivalent moment factor with respect to literature, which could be the result of the conservative nature of general simplified design expressions. For  $\psi < 0$ , the ratio increases significantly as a result of the assumed symmetric sinusoidal buckling shape not being accurate for very asymmetric loading. This discrepancy for asymmetric load patterns is shown by Fig. 5, in which the finite difference results of Serna et al. for an IPE500

steel beam with length  $L = 8m$  are elaborated, with parameters chosen to represent practical usual cases (although differences with other lengths and profiles are small).

For a beam loaded by two equal end-moments or one end-moment and a concentrated mid-span load (Figs. 6 and 7), the general formulation shows strong similarity with the finite difference results of Serna et al. for both free and restrained warping. For the case with one end-moment differences up to 13.8% are obtained, following from the introduced asymmetry of the loading for large  $\alpha$ , causing deviations from the assumed symmetric buckling shapes. Figs. 8 and 9 compare the general formulation with literature for an uniformly distributed load with one or two (equal) end-moments and it is found that, for this load cases, both the Australian Standard AS-4100 and Eq. (37) show good correspondence with the finite difference results of Serna et al. [20] and Suryatmono and Ho [22] with differences up to 6.7% and 12.7% respectively. The available design standards and literature are captured with reasonably accuracy with the aid of Eqs. (34) and (37), depending on the complexity and asymmetry of the considered loading pattern. Hence, when making use of the general formulation, it is important to verify whether the assumed buckling shapes (Eqs. (33) and (35)) are sufficiently accurate for the purpose of the analysis. Instead of case-specific (possibly more accurate) analysis, this study offers a general formulation that can be used to consider all loading cases within the formulated theoretical framework with no significant loss of accuracy.

## 7. Application

With the aid of the general formulations for point loads (Eq. (34)) and uniformly distributed loads (Eq. (37)), some specific cases are analyzed and compared with the design standards to show the application of the theory.

### 7.1. Elaboration of example

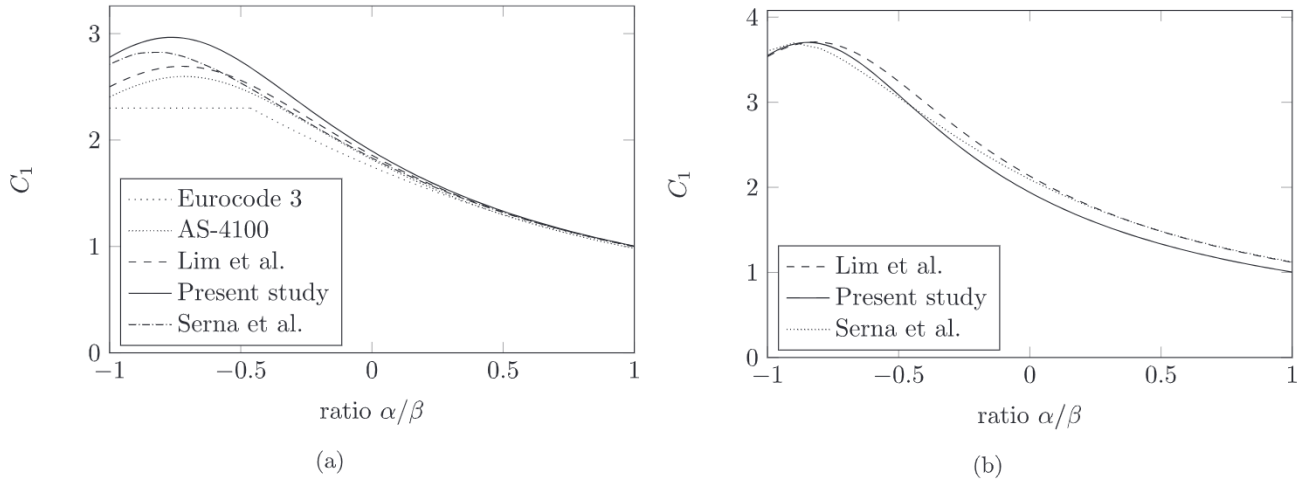
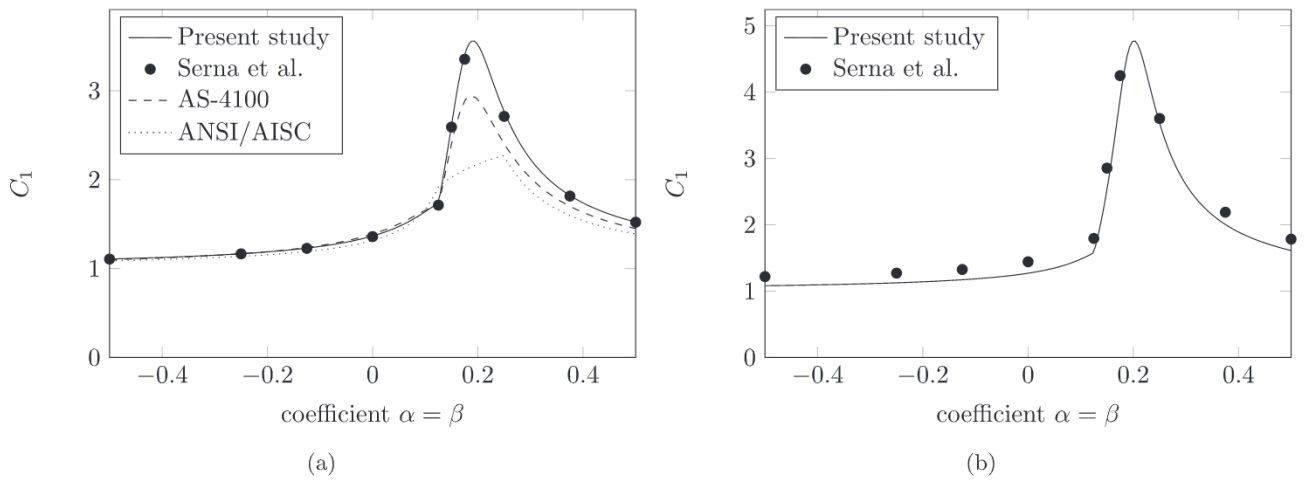
A simply supported beam with length  $L = 8m$  without warping restraints ( $k = k_w = 1$ ), loaded at 1/2 of the span by a point load  $F = 12.5kN$  is considered. The beam is loaded by support-moments  $M_A = M_B = 17.5kNm$ , leading to  $\alpha = \beta = M_A/FL = 0.175$ . For a beam

**Table 4**Coefficients  $D_3$ ,  $D_4$  and  $D_5$  multiple point loads at  $\gamma_i L$  with and without warping allowed at the supports.

$\gamma_i$	UDL	[0.25, 0.75]	[0.33, 0.67]	[0.25, 0.50]	[0.25, 0.50, 0.75]	[0.33, 0.50, 0.67]	[0.33, 0.33, 0.67]
$k_w = 1$							
$D_3$	-0.1086	-0.2382	-0.2962	-0.3069	-0.4138	-0.4718	-0.4554
$D_4$	-0.1086	-0.2382	-0.2962	-0.2826	-0.4138	-0.4718	-0.4331
$D_5$	0.0122	0.0578	0.0912	0.0923	0.1774	0.2341	0.2066
$k_w = 0.5$							
$D_3$	-0.1150	-0.2466	-0.3147	-0.3247	-0.4733	-0.5073	-0.4811
$D_4$	-0.1150	-0.2466	-0.3147	-0.3069	-0.4733	-0.5073	-0.4630
$D_5$	0.0134	0.0611	0.1004	0.1029	0.2279	0.2628	0.2270

**Table 5**Comparison of the equivalent moment factors  $C_1$  with the general formulation of this study for several standard load cases and free warping ( $k = k_w = 1$ ).

Load case	Present	EC3	AS-4100	ANSI	Trahair	Serna	(1)/(2)	(1)/(3)	(1)/(4)	(1)/(5)	(1)/(6)
	(1)	(2)	(3)	(4)	(5)	(6)					
$\psi = 1$	1.000	1.000	0.982	1.000	1.000	1.000	1.00	1.02	1.00	1.00	1.00
$\psi = 0.75$	1.141	1.131	1.120	1.111	1.131	1.137	1.01	1.02	1.03	1.01	1.00
$\psi = 0.5$	1.324	1.300	1.297	1.250	1.300	1.310	1.02	1.02	1.06	1.02	1.01
$\psi = 0.25$	1.564	1.506	1.525	1.429	1.506	1.532	1.04	1.03	1.09	1.04	1.02
$\psi = 0$	1.881	1.750	1.818	1.667	1.750	1.815	1.07	1.03	1.13	1.07	1.04
$\psi = -0.25$	2.284	2.031	2.164	2.000	2.031	2.155	1.12	1.06	1.14	1.12	1.06
$\psi = -0.5$	2.712	2.300	2.483	2.174	2.350	2.488	1.18	1.09	1.25	1.15	1.09
$\psi = -0.75$	2.940	2.300	2.594	2.222	2.500	2.652	1.28	1.13	1.32	1.18	1.11
$\psi = -1$	2.766	2.300	2.404	2.273	2.500	2.523	1.20	1.15	1.22	1.11	1.10
UDL	1.133	1.130	1.166	1.136	1.130	1.136	1.00	0.97	1.00	1.00	1.00
Mid PL	1.366	1.350	1.388	1.316	1.350	1.276	1.01	0.98	1.04	1.01	1.07
Quarter PL	1.494	1.450	1.363	1.364	1.450	1.390	1.03	1.10	1.10	1.03	1.07
2 PL (0.25, 0.75)	1.040	1.040	0.982	1.000	1.088	1.000	1.00	1.06	1.04	0.96	1.04
Mid PL SC	1.830	1.700	1.817	1.705	1.800	1.596	1.08	1.01	1.07	1.02	1.15
Mid PL DC	1.732	1.640	1.700	1.923	1.710	1.435	1.06	1.02	0.90	1.01	1.21
UDL SC	2.276	2.200	2.404	2.083	2.250	2.197	1.03	0.95	1.09	1.01	1.04
UDL DC	2.613	2.300	2.500	2.381	2.420	2.574	1.14	1.05	1.10	1.08	1.02
Average							1.07	1.04	1.09	1.05	1.06
Standard dev.							0.08	0.05	0.10	0.06	0.06

**Fig. 5.**  $C_1$  as function of the ratio  $\alpha/\beta$  (or coefficient  $\psi$  in Eurocode 3) for (a) unrestrained warping with  $k_w = 1$  and (b) restrained warping with  $k_w = 0.5$ .**Fig. 6.**  $C_1$  for a beam loaded by two equal end-moments and a concentrated mid-span load for (a) unrestrained warping with  $k_w = 1$  and (b) restrained warping with  $k_w = 0.5$ .



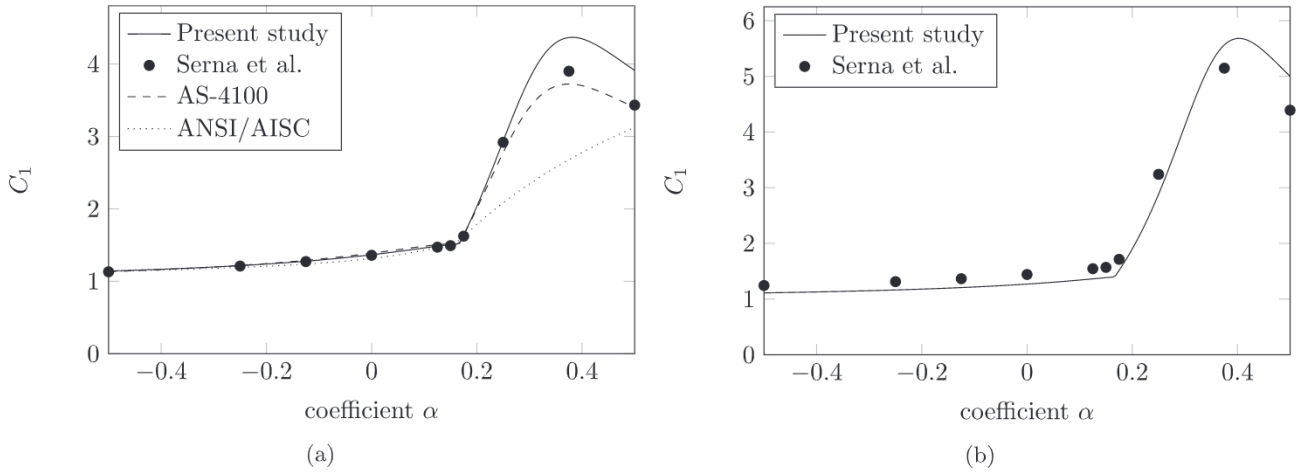


Fig. 7.  $C_1$  for a beam loaded by one end-moment ( $\beta = 0$ ) and a concentrated mid-span load for (a) unrestrained warping with  $k_w = 1$  and (b) restrained warping with  $k_w = 0.5$ .

loaded at  $1/2$  of the span, Table 3 gives  $D_3 = -0.1757$ ,  $D_4 = -0.1757$  and  $D_5 = 0.0335$  and from Appendix A follows  $D_1 = 0.2827$  and  $D_2 = 0.4347$ . The maximum moment  $M_{max} = 17.5kNm$  is found at the supports. From Eq. (34), the equivalent moment factor  $C_1 = 3.405$ . From the diagrams of Eurocode 3 for concentrated loading and support-moments follows  $C_{1,EC3} = 2.30$ . Furthermore,  $C_{1,ANSI} = 2.886$  and  $C_{1,AS-4100} = 2.108$  are found based on Eqs. (6) and (5). Using the numerical finite difference approach, Serna et al. found  $C_{1,Serna} = 3.354$  for this case, which is assumed to be (close to) the exact value. Hence, when using the design standards, the structure is over-conservative by a factor 1.18 for ANSI and even 1.62 for AS-4100, leading to inefficient use of the cross-section.

When considering the same beam loaded at  $L/3$  with  $\alpha = 0.175$  and  $\beta = 0.035$ , the general formulation for point loads results in  $C_1 = 3.083$ . For non-midspan loadings, Eurocode 3 does not provide a solution. Assuming that a mid-span load is a proper representation of the considered case, Eurocode 3 provides  $C_{1,EC3} = 1.60$ . The American Steel Standard gives  $C_{1,ANSI} = 2.720$  and the Australian Steel Standard provides  $C_{1,AS-4100} = 4.344$ , which is 1.6 times larger than  $C_{1,ANSI}$ . The inconsistency between different design standards is the result of the non-physical nature of simplified expressions created by curve-fitting, causing specific cases, with for example  $M_a = M_c = 0$ , to deviate significantly from the physical reality. The general formulation of Eq. (34) is based on physical considerations and is therefore less likely to show these inconsistencies.

## 7.2. Combination of PL and UDL

General formulations are given for cases with only point loads or UDL. However, it is also possible to consider the combination of PL and UDL by representing the UDL by a set of  $n$  equivalent equally-spaced point loads and add these locations to  $\gamma_i$  to make use of Eq. (37) (although it is also possible to solve Eq. (31) including the work done by the UDL and the corresponding  $M_y$  instead of using the general formulation). Consider a simply supported beam loaded by a mid-span point load  $F$  and an UDL  $q = \theta F/L$ . The UDL is represented by  $n$  equivalent unit loads  $F_i = qL/n$ , where it is advised to use  $n > 10$  to obtain a proper representation of the UDL. The concentrated load is represented by  $n/\theta$  unit point loads  $F_i$ . The collection of locations  $\gamma_i$  is now found by

$$\gamma_i = \left[ \frac{2j-1}{2n}L; \gamma_{F,k} \right] \quad \text{for } j = 1 \dots n \quad (38)$$

with  $k = n/\theta$  and  $\gamma_{F,k} = \gamma_F$ , where  $n$  is chosen such that  $k$  is an integer. Following this approach, it has been found that the equivalent moment factor for the combination of a mid-span point load and an UDL can be accurately described by

$$C_1 = \begin{cases} 1.27 \cdot \theta^{-0.034} & \text{for } \theta \leq 25 \\ 1.13 & \text{for } \theta > 25 \end{cases} \quad (39)$$

Using the general formulations of this study, design figures and diagrams can be straightforwardly created. For this purpose, it is

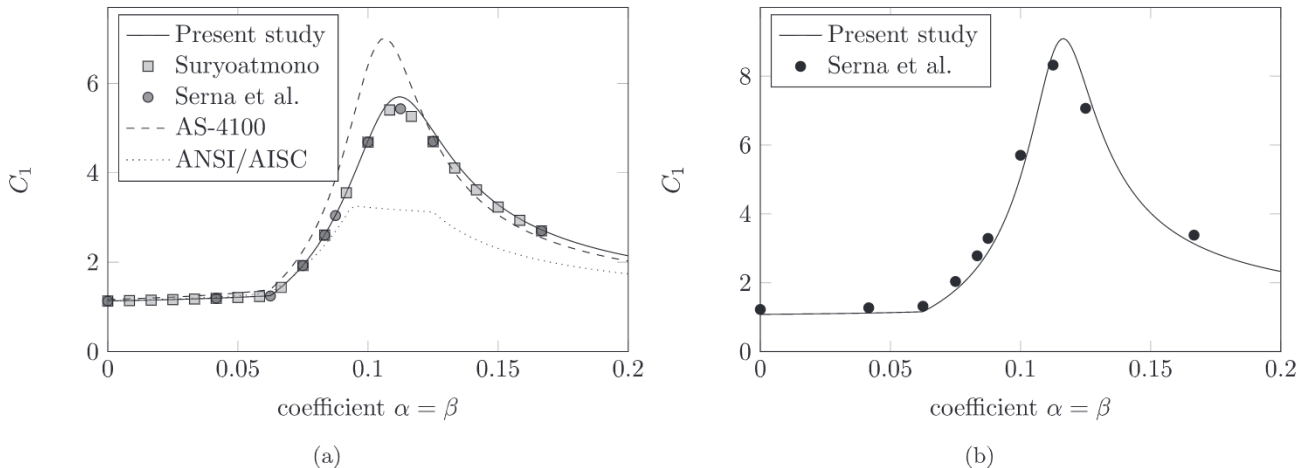


Fig. 8.  $C_1$  for a beam loaded by two end-moments and a uniformly distributed load for (a) unrestrained warping with  $k_w = 1$  and (b) restrained warping with  $k_w = 0.5$ .

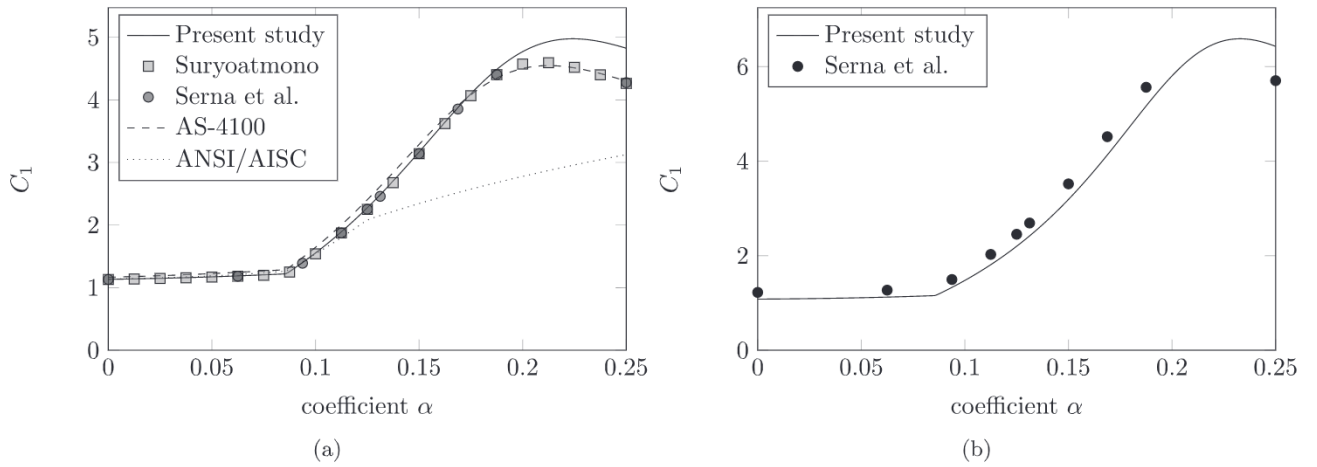


Fig. 9.  $C_1$  for a beam loaded by one end-moment ( $\beta = 0$ ) and a uniformly distributed load for (a) unrestrained warping with  $k_w = 1$  and (b) restrained warping with  $k_w = 0.5$ .

advised to program the general formulations and expressions of Appendix A. Furthermore, it is also possible to consider multiple uniformly distributed loads over parts of a span by changing the distance between the equivalent point loads for each UDL.

Lastly, a rather complex but academic example is considered, as depicted by Fig. 10, in order to exemplify the application of the general formulation to combined point loads and uniformly distributed loads and illustrate the wide range of possible load conditions that can be captured within the general formulation. The beam is simply supported without warping restraints ( $k = k_w = 1$ ) and the uniformly distributed load only works over a minor part of the beam. Following the proposed method,  $C_1 = 2.378$  is obtained. As the general formulation is based on physical considerations, the obtained equivalent moment factor is probably more physically justified compared to general simplified design equations. For example the Australian Standard results in  $C_{1,AS-4100} = 2.822$ , which is 18.6% larger than the equivalent moment factor following from the proposed general formulation. Due to its complexity, this example can not be analyzed using the Eurocode.

### 7.3. Intermediate torsional/lateral supports

This study considers single-span beams without intermediate supports. To increase the resistance to lateral torsional buckling, intermediate torsional and/or lateral supports can be added. A first indication of the equivalent moment factor can be achieved by assuming that each segment, defined as the part between two supports, behaves as an independent individual segment. This assumption holds for similar segments that have similar resistances. For non-similar segments however, the assumption of negligible interaction does not hold anymore since the stronger segments restrain the weaker ones and a more elaborate analysis should be performed. The same effects are obtained for

multi-span beams over multiple supports. These effects are not further elaborated in this study.

### 7.4. Approach within context of design procedure

With the aid of the derived general formulation, the equivalent moment factor for single span beams without intermediate restraints is determined for both I-section (often steel) and rectangular slender sections (often timber). The determination of the critical theoretical (Euler) buckling load is a general physical problem, which does not depend on the considered material. In the regular design procedures, the theoretical buckling load, as determined with the equivalent moment factor (Eqs. (1) and (2)), is reduced in order to incorporate effects of material imperfections, geometric imperfections, internal stresses etc. The reduction of the theoretical buckling load is a material dependent procedure and is therefore not further considered in this study, but is mentioned here to capture the complete design procedure. Furthermore, for general application of the derived general formulation, the influence of the loading location within the cross-section on the buckling load should be studied in more detail.

## 8. Conclusions

Lateral torsional buckling is considered as an ultimate limit state failure mode during the design of beam structures. This study focused on the influence of the moment distribution via the equivalent moment factor  $C_1$  on the critical elastic moment for both I-sections (often steel) and slender rectangular sections (often timber). The considered design standards offer different simplified general solutions and some practical case-specific solutions. The advantage of simplicity comes with the disadvantage of loss of accuracy. Furthermore, the goal of design standards is to be conservative rather than to be accurate. On the other side, literature offers many case-specific studies that go in depth on a certain load case, mostly leading to an accurate, but complex, solution.

This paper has presented a general formulation that is in between the simplicity of the design standards and the specific complex solution procedures as can be found in literature. This general formulation is derived based on the principle of conservation of energy and can be used for support-moments, point loads (Eq. (34)), uniformly distributed loads (Eq. (37)) and a combination of those, for free warping ( $k = k_w = 1$ ) and restrained warping ( $k = 1$  and  $k_w = 0.5$ ) at the supports. For the case of free warping, coefficients  $D_1$  to  $D_5$  are defined in Appendix A. The results of this paper have been compared with design standards (EC3 [4], AS-4100 [6], ANSI/AISC 360-17 [7]) as well as literature (Trahair [8], Serna et al. [20], Lim et al. [19] and Suryoatmono and Ho [22]) and it is found that this paper is in good

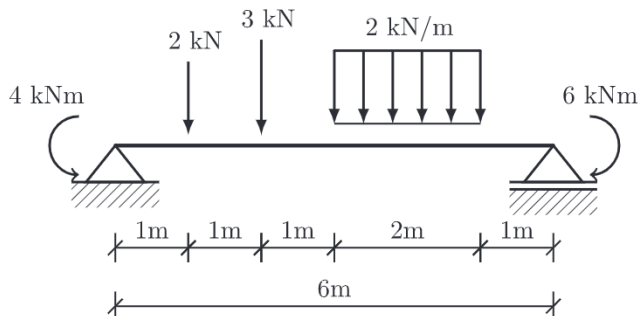


Fig. 10. Simply supported beam loaded by point loads, uniformly distributed load and end-moments.

correspondence with design standards and literature. On average, the ratio between the results of this study and literature lies between 1.04 and 1.09. The expected accuracy of the general formulation decreases for asymmetric loading as a result of the assumed symmetric buckling shapes (Eqs. (33) and (35)) not representing the physical reality. Furthermore, it is noted that the general formulation of this study is not exact as it is based on approximate buckling shapes. However, the quality of the obtained equivalent moment factors has been properly assessed through comparisons with other studies and design standards. The influence of length, warping and torsion stiffness on  $C_1$  for restrained warping ( $k_w = 0.5$ ) is neglected in this study, inducing a potential loss of accuracy.

The application of the general formulations has been shown with the aid of practical examples, pointing out possible drawbacks of using design standards. By manipulating the collection of locations  $\gamma_i$ , it is possible to represent complex load-cases of multiple UDL and point loads combined. Instead of using the general formulation, it is also possible to make use of Eq. (31). The general formulation presented in

#### Appendix A. . Expressions $D_1$ to $D_5$ for $k = k_w = 1$

$$D_1 = \frac{1}{3} - \frac{1}{2\pi^2} \approx 0.2827 \quad (\text{A.1})$$

$$D_2 = \frac{1}{3} + \frac{1}{\pi^2} \approx 0.4347 \quad (\text{A.2})$$

$$D_3 = \sum_{i=1}^N \left\{ -(\gamma_i - 1)P_i + Q_i + R_i + \frac{\gamma_i - 1}{\pi^2} \right\} \quad (\text{A.3})$$

$$D_4 = \sum_{i=1}^N \left\{ \gamma_i P_i - Q_i + \frac{\gamma_i^3 - \gamma_i}{3} - \frac{\gamma_i}{\pi^2} \right\} \quad (\text{A.4})$$

$$D_5 = \sum_{i=1}^N \left\{ P_i S_i - Q_i \left[ \sum_{j=1}^N \gamma_j - T_i \right] + \frac{1}{3} (\gamma_i^4 + (U_i - [n + 2 - i])\gamma_i^3 + \gamma_i^2) - \sum_{j=i+1 \leq N}^N R_j \gamma_i + \frac{1}{\pi^2} \left( -\frac{1}{2} \gamma_i^2 + (T_i - U_i) \gamma_i \right) \right\} \quad (\text{A.5})$$

with

$$\begin{aligned} P_i &= \frac{1}{\pi^2} \cos^2 \pi \gamma_i \\ Q_i &= \frac{1}{\pi^3} \sin \pi \gamma_i \cos \pi \gamma_i \\ R_i &= \frac{-\gamma_i^3 + 3\gamma_i^2 - 2\gamma_i}{3} \\ S_i &= \sum_{j=1}^{i-1} [\gamma_j (\gamma_i - 1)] + \sum_{j=i}^N [\gamma_i (\gamma_j - 1)] \\ T_i &= \frac{1 + 2(N - i)}{2} \\ U_i &= \sum_{j=i+1 \leq N}^N \gamma_j \\ 0 &\leq \gamma_i \leq 1 \end{aligned} \quad (\text{A.6})$$

and the maximum moment

$$M_{\max} = \max(|M_{y, \gamma_i}|) \quad (\text{A.7})$$

with

$$M_{y, \gamma_i} = FL \left[ \sum_{j=1}^{i-1} \gamma_j + \gamma_i (N - i + 1 - \sum_{j=1}^N \gamma_j) - \alpha(1 - \gamma_i) - \beta \gamma_i \right] \quad (\text{A.8})$$

#### References

- [1] Lindner J. Design of steel beams and beam columns. Eng Struct 1997;19(5):378–84. stability of structures.
- [2] Pi YL, Trahair NS. Prebuckling deflections and lateral buckling. i: Theory. J Struct Eng 1992;118(11):2949–66.
- [3] Timoshenko SP, Gere JM. Theory of elastic stability. New York: McGraw-Hill; 1961.
- [4] European Committee for Standardization. EC3 2016. EN 1993- 1-1. Eurocode 3: Design of steel structures. Part 1-1: General rules and rules for buildings, Brussels,

this paper is limited to elastic material behaviour, in plane loading, loading in the shear center and double-symmetric cross-sections and can not consider the influence of axial loading. In order to work with the general formulation, one must be aware of these limitations.

In a more general picture, this paper contributes to the solution of Sahraei et al. [15], which takes account of partial twist restraints, load height, moment distribution and pre-buckling deformations. In their study, Sahraei et al. made use of the Australian Steel Standards to consider the influence of the moment distribution. With the contribution of this paper, the theory of Sahraei et al. is improved and more physical background is obtained.

#### Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

- Belgium, 2016.
- [5] European Committee for Standardization. EC5 2011. EN 1995- 1-1. Eurocode 5: Design of timber structures. Part 1-1: General rules and rules for buildings, Brussels, Belgium, 2011.
- [6] Standards Australia. AS-4100 1998. AS-4100 Steel Structures, Sydney, Australia, 1998.
- [7] American Institute of Steel Construction (AISC). ANSI/AISC 360–16. Specification for Structural Steel Buildings. Chicago, USA: AISC; 2016.
- [8] Trahair NS. Flexural-torsional Buckling of Structures. London: E & FN Spon; 1993.
- [9] Chajes A. Principles of Structural Stability Theory. New Jersey, USA: Prentice Hall



- Inc; 1974.
- [10] Gardner L, Nethercot DA, Gholvanessian H. *Designers Guide to Eurocode 3: Design of Steel Buildings*. 2nd ed. ICE Publishing; 2011.
  - [11] Bijlaard FSK, Steenbergen HMGM. Lateral torsional stability of members with lateral restraints at various locations between the supports, p. 191–200.
  - [12] American Wood Council. AFPA-TR14. Designing for lateral-torsional stability in wood members. AFPA-Technical Report 14, Washington, DC, USA: American Forest Paper Association, Inc.; 2003.
  - [13] Kirby PA, Nethercot DA. *Design for Structural Stability*. Australia: John Wiley Sons; 1979.
  - [14] Canadian Institute of Steel Construction. CAN-CSA S16-14. Handbook of steel construction. CSA-S16-14 specification. Canada: Standards Council of Canada; 2014.
  - [15] Sahraei A, Pezeshky P, Mohareb M, Doudak G. Simplified expressions for elastic lateral torsional buckling of wooden beams. *Eng Struct* 2018;174:229–41.
  - [16] Pi YL, Trahair NS, Rajasekaran S. Energy equation for beam lateral buckling. *J Struct Eng* 1992;118(6):1462–79.
  - [17] Xiao Q, Doudak G, Mohareb M. Numerical and experimental investigation of lateral torsional buckling of wood beams. *Eng Struct* 2017;151:85–92.
  - [18] Burow J, Manbeck H, Janowiak J. Lateral stability of composite wood i-joists under concentrated-load bending. *Trans ASABE* 2006;49(6):1867–80.
  - [19] Lim N, Park N, Kang Y, Sung I. Elastic buckling of i-beams under linear moment gradient. *Int J Solids Struct* 2003;40(21):5635–47.
  - [20] Serna M, Lopez A, Puente I, Yong D. Equivalent uniform moment factors for lateral-torsional buckling of steel members. *J Const Steel Res* 2006;62:566–80.
  - [21] Nethercot DA, Rockey KC. Unified approach to the elastic lateral buckling of beams. *Eng J-Am Instit Steel Constr* 1972;9(3):96–107.
  - [22] Suryatmono B, Ho D. The moment gradient factor in lateral-torsional buckling on wide flange steel sections. *J Constr Steel Res* 2002;58(9):1247–64.
  - [23] Yoo CH, Lee SC. Chapter 6 - torsional and flexural-torsional buckling. In: Yoo CH, Lee SC, editors. *Stability of Structures*. Boston: Butterworth-Heinemann; 2011. p. 303–26.