Buckling load of non-uniform columns

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A 20 m column consists of three steel tubular sections welded together. The middle part is 17 m long, has a diameter of 1036 mm and a wall thickness of 50 mm. The end parts are 1.5 m long, have diameters of 1016 mm and wall thicknesses of 40 mm. Young's modulus of steel is $2.1 \cdot 10^5$ N/mm²

The cross-section properties are

$$I = \pi R^3 t$$

$$I_m = \pi \left(\tfrac{1}{2}1036\right)^3 50 = 2183 \cdot 10^7 \text{ mm}^4$$

$$I_e = \pi \left(\tfrac{1}{2}1016\right)^3 40 = 1647 \cdot 10^7 \text{ mm}^4.$$

Since an exact formula for the buckling load does not exist, it is approximated using potential energy.

If the column is pinned at both ends the deflection *w* can be approximated by

$$w = \hat{w} \sin \frac{\pi x}{l}.$$

If the column is fixed at both ends the deflection *w* can be approximated by

$$w = \hat{w} \tfrac{1}{2}\left(1 - \cos \frac{2\pi x}{l}\right),$$

where $\hat{w}$ is the maximum deflection in the middle of the column.

The curvature κ is

$$\kappa = -\frac{d^2 w}{dx^2}.$$

The moment *M* in a section is

$$M = EI\kappa.$$

The potential energy E_{pot} of the column is

$$E_{pot} = \int_0^l \tfrac{1}{2} M \kappa \, dx - P u \,,$$

where P is the axial loading and u is the axial displacement.
The axial displacement can be calculated with

$$u = l - \int_0^l \sqrt{1 - (\frac{dy}{dx})^2}\, dx \,.$$

The latter formula can be derived by considering an infinitesimal part of the deformed column. However, it cannot be evaluated easily. Using the Taylor series $\sqrt{1-f} = 1 - \tfrac{1}{2} f + \mathrm{O}(f^2)$ we can approximate u accurately.

$$u \approx \tfrac{1}{2} \int_0^l (\frac{dy}{dx})^2 \, dx \,.$$

Evaluation of the E_{pot} equation gives

$$E_{pot} = \int_0^{1500} \tfrac{1}{2} M_e \kappa \, dx + \int_{1500}^{18500} \tfrac{1}{2} M_m \kappa \, dx + \int_{18500}^{20000} \tfrac{1}{2} M_e \kappa \, dx - P u \,.$$

Evaluation of for pinned edges gives

$$E_{pot} = (13937 - 0.0001234\, P)\, \hat{w}^2 \,.$$

Evaluation for clamped edges gives

$$E_{pot} = (52006 - 0.0001234\, P)\, \hat{w}^2 \,.$$

The column is stable when the potential energy curve (E_{pot} versus $\hat{w}$) is bowl shaped. It becomes unstable when P is such that the bottom of the “bowl” is flat. Therefore, it is unstable when

$$\frac{d^2 E_{pot}}{d\hat{w}^2} = 0 \,.$$

So, for the pinned column

$$2(13937 - 0.0001234\,P_{cr}) = 0$$

$$P_{cr} = \frac{13937}{0.0001234} = 113000 \cdot 10^3 \text{ N.}$$

and for the fixed column

$$2(52006 - 0.0001234\,P_{cr}) = 0$$

$$P_{cr} = \frac{52006}{0.0001234} = 421000 \cdot 10^3 \text{ N.}$$

The pinned end result has been checked with a structural analysis program (geometrical nonlinear analysis with Dr. Frame including a small lateral load). P_{cr} = 112000 kN, which is almost the same.

The method gives good results when the estimated buckling shape agrees well with the real buckling shape. If not, the method will overestimate the real buckling load.

This method works also for a cross-section that changes gradually over the length (tapered sections).