Chapter 16
Reinforcement Design Using Linear Analysis

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Design of reinforced concrete structures can be described by the following consecutive steps:

1. Select the initial dimensions of all the structural elements using simple rules of thumb or experience. These dimensions should be able to satisfy the serviceability and ultimate limit states, and should fulfill the requirements for adequate site execution and any other applicable requirement (e.g. acoustic isolation, fire protection, etc.).
2. Perform a global structural analysis to calculate the internal forces (or stresses) due to the combination of loads defined in the codes. The method almost used exclusively today is the finite element method and the behaviour of the structure is assumed to be linear elastic at this stage.
3. Verify concrete initial dimensions and calculate the reinforcement capable of resisting the calculated internal forces. At this stage, the ultimate capacity of the individual cross-sections is considered; this which is typically associated with nonlinear constitutive laws.

The main advantage of the above process is that linear elastic finite element analysis is well established and is straightforward to apply. In addition, multiple load cases can be easily incorporated and reinforcement is placed in the locations where tensile stresses appear. These regions correspond to the initial crack locations, helping to control crack propagation.

Of course there are also some disadvantages in the process described: stress redistribution can be difficult to incorporate, providing more expensive reinforcement arrangement; no real information is obtained about the collapse load of the structure, even if a lower bound estimate is obtained when ductility is enforced, and no real information is provided on inelastic

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phenomena as crack width, crack spacing or maximum deflection, even if they can be estimated for beam-type structural elements. A consequence of the process is that detailing guidelines need to be used to ensure ductility and serviceability demands.

Only in very few selected cases of structures with unusual size, shape, or complexity, would a full nonlinear analysis of the previously designed structure be made for assessment, tracing out the entire behaviour through the uncracked, cracked, and ultimate stages. Such an analysis generally requires significant time for pre-processing, computation and post-processing, which is not compatible with cost and time demands. Also, as nonlinear analysis requires the definition of geometry and reinforcement, it should not be regarded as a design tool but, mostly, as an assessment tool.

The simulation of concrete walls, slabs, assemblages of walls and slabs and shells using finite element analysis is becoming a standard in structural analysis tools for building design, meaning that adequate methodologies for the design of these elements are necessary.

In this chapter, we discuss design methods for membrane states (walls), bending states (slabs) and combinations as may occur in spatial assemblages of plates and in shells. Hereafter they are referred to as shells. Design of reinforced concrete elements subjected to membrane states has been developed since 1960s by authors like Baumann, Braestrup and Nielsen, only to name of few. The book of Nielsen is a classical reference [19]. This process resulted in formulas for reinforcement design and check of concrete strength in the CEB-FIB Model Code 1990 for Concrete Structures [20].

Reinforcement design for slabs and shells has also received attention in the Model Code 1990. For that purpose a three layer sandwich model was introduced. Pioneers of this approach are Gupta and Marti. The introductory version of Eurocode 2 suggested a different method based on the normal yield criterion. That result is alternatively referred to as the Wood–Armer equations [21]. It applies to slabs only and not for shells. The later version EN 1992-1-1:2004 of Eurocode 2 [22] removed this method. The version EN 1992-2:2005 of Eurocode 2 again included a solution in its appendices, returning to the three layer sandwich model. This solution has received a place in Part 2 of Eurocode 2 on Bridges [23]. *fib* has published in 2008 a *Practitioners' guide to finite element modeling of reinforced concrete structures* [24]. This document also presents the three layer sandwich model. Readers interested in a more complete review of the historical development of the different methods are referred to [25, 26].

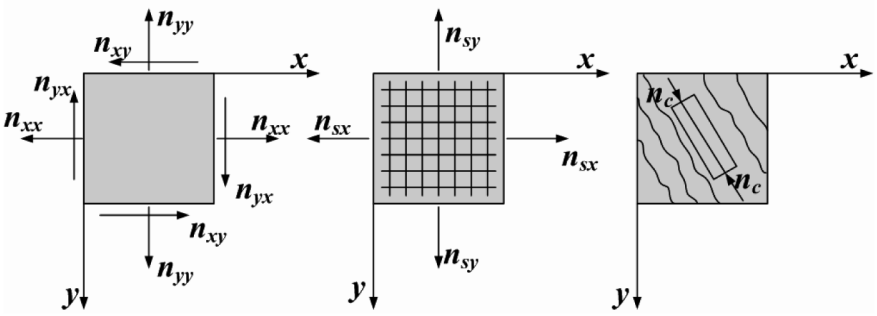

Figure 16.1 Model for membrane element: (a) applied forces; (b) reinforcement contribution; (c) concrete contribution.

Here we will refer to the three layer sandwich model of Eurocode 2 and *fib* practitioners' guide as *basic model*. We call it *basic* because the concept is very useful, but the working-out still requires improvement, because internal lever arms are only approximated and some terms are neglected in the equilibrium equations. After the presentation of the basic model, we introduce an *advanced method* which is consistent with all terms in the equilibrium conditions.

Both the basic and advanced model will first be discussed for cases of moderate transverse shear forces which can be carried by the concrete. After that we make the extension to slabs with larger transverse shear forces; these require transverse shear reinforcement, an extension we owe to Marti [27].

16.1 Design of Membrane States

Consider a membrane element with a thickness h, subjected to applied in-plane forces n_{xx}, n_{yy} and n_{xy}, as shown in Figure 16.1a. The reinforcement consists of two orthogonal sets of rebars parallel to the x, y-axes. a_{sx} and a_{sy} are the needed reinforcement areas per unit length in this co-ordinate system. They are calculated from forces n_{sx} and n_{sy} respectively. The purpose of this section is to find formulas for n_{sx} and n_{sy}.

The applied forces will be resisted by the reinforcement and concrete contributions. It is assumed that the concrete is subjected to uni-axial compression n_c parallel to the cracking orientation, at an angle θ with the y-axis. The two rebar sets in Figure 16.1b and the concrete struts of Figure 16.1c must together carry the applied loads of Figure 16.1a. For the sign convention of the applied n_{xx}, n_{yy} and n_{xy} loads we refer to Chapter 1. The forces n_{sx} and

n_{sy} are always positive or zero, and the membrane force n_c in the concrete is negative or zero.

In the chosen x, y-coordinate system, the shear resistance of the reinforcement is zero and the state of stress of the concrete is uni-axial. The first principal membrane force is zero and the compressive force n_c occurs in the second principal direction. The stress state in Figure 16.1a is equivalent to the combination of the states in Figures 16.1b and 16.1c when the following equilibrium conditions are satisfied:

$$\begin{aligned} n_{xx} &= n_{sx} + n_c \sin^2\theta \\ n_{yy} &= n_{sy} + n_c \cos^2\theta \\ n_{xy} &= -n_c \sin\theta \cos\theta \end{aligned} \tag{16.1}$$

The second principal stress σ_c must be smaller than the compressive strength f_c of concrete

$$n_c \geq -hf_c \tag{16.2}$$

The applied forces are in the left member of Eq. (16.1) and the internal forces are in the right member. It should be remembered that n_c is negative, so n_{sx} is not smaller than n_{xx} and n_{sy} is not smaller than n_{yy}. The cases of $\theta = 0$ and $\theta = \pi/2$ are trivial, meaning that the principal directions are aligned with axes x and y. If $\theta \neq 0$ and $\theta \neq \pi/2$, Eqs. (16.1) to (16.2) can be recast such that the steel and concrete forces are in the left member and the applied forces are in the right:

$$\begin{aligned} n_{sx} &= n_{xx} + n_{xy} \tan\theta \\ n_{sy} &= n_{yy} + n_{xy} \cot\theta \\ n_c &= -\frac{n_{xy}}{\sin\theta \cos\theta} \end{aligned} \tag{16.3}$$

The third expression in Eq. (16.3) indicates that n_{xy} and θ must have the same sign, so that n_c is negative, or in compression. The total amount of reinforcement can be obtained from the first two expressions in Eq. (16.3), and is

$$n_{sx} + n_{sy} = n_{xx} + n_{yy} + n_{xy}(\tan\theta + \cot\theta) \tag{16.4}$$

Note that the last term in this equation is always positive, as n_{xy} and θ have the same sign. Thus, the minimum amount of reinforcement corresponds to $\theta = \pm\pi/4$. For these values of θ, noting that the reinforcement must be always subjected to tension, i.e. $n_{sx} \geq 0$ and $n_{sy} \geq 0$, the first two expressions of Eq. (16.3) gives $n_{xx} \geq -|n_{xy}|$ and $n_{yy} \geq |-n_{xy}|$ respectively. Otherwise the θ value must be changed. Therefore four different cases of reinforcement have to be considered.

Case 1: Reinforcement in X- and Y-Direction Needed

For this case it holds

$$\begin{aligned} &n_{xx} \geq -|n_{xy}|, && n_{yy} \geq -|n_{xy}| \\ &n_{sx} = n_{xx} + |n_{xy}|, && n_{sy} = n_{yy} + |n_{xy}| \\ &\theta = \pm\frac{\pi}{4}, && n_c = -2|n_{xy}| \end{aligned} \tag{16.5}$$

Case 2: Only Reinforcement in Y-Direction Needed

For this case the following equations hold:

$$\begin{aligned} &n_{xx} < -|n_{xy}| \rightarrow n_{sx} = 0 \\ &\tan\theta = -\frac{n_{xx}}{n_{xy}} \\ &n_{sy} = n_{yy} - \frac{n_{xy}^2}{n_{xx}}, \quad n_{sy} \geq 0 \rightarrow n_{yy} \geq \frac{n_{xy}^2}{n_{xx}} \\ &n_c = n_{xx} + \frac{n_{xy}^2}{n_{xx}} \end{aligned} \tag{16.6}$$

Case 3: Only Reinforcement in X-Direction Needed

For this case the following equations hold:

$$\begin{aligned} &n_{yy} < -|n_{xy}| \rightarrow n_{sy} = 0 \\ &\tan\theta = -\frac{n_{xy}}{n_{yy}} \\ &n_{sx} = n_{xx} - \frac{n_{xy}^2}{n_{yy}}, \quad n_{sx} \geq 0 \rightarrow n_{xx} \geq \frac{n_{xy}^2}{n_{yy}} \\ &n_c = n_{yy} + \frac{n_{xy}^2}{n_{yy}} \end{aligned} \tag{16.7}$$

Case 4: No Reinforcement Needed in Any Direction

No cracking occurs and the stress state is biaxial compression. In the concrete two principal membrane forces n_{c1} and n_{c2} are present:

Case	n_{sx}	n_{sy}	n_c
$n_{xx} \geq -\lvert n_{xy}\rvert$ $n_{yy} \geq -\lvert n_{xy}\rvert$	$n_{xx} + \lvert n_{xy}\rvert$	$n_{yy} + \lvert n_{xy}\rvert$	$-2\lvert n_{xy}\rvert$
$n_{xx} < -\lvert n_{xy}\rvert$ $n_{yy} \geq \dfrac{n_{xy}^2}{n_{xx}}$	0	$n_{yy} - \dfrac{n_{xy}^2}{n_{xx}}$	$n_{xx} + \dfrac{n_{xy}^2}{n_{xx}}$
$n_{xx} \geq \dfrac{n_{xy}^2}{n_{yy}}$ $n_{yy} < -\lvert n_{xy}\rvert$	$n_{xx} - \dfrac{n_{xy}^2}{n_{yy}}$	0	$n_{yy} + \dfrac{n_{xy}^2}{n_{yy}}$
other	0	0	$\dfrac{n_{xx}+n_{yy}}{2} - \sqrt{\left(\dfrac{n_{xx}+n_{yy}}{2}\right)^2 + n_{xy}^2}$
$a_{sx} = \dfrac{n_{sx}}{f_{syd}}$		$a_{sy} = \dfrac{n_{sy}}{f_{syd}}$	$f_c = \dfrac{n_c}{h} \leq f_{cd}$

Figure 16.2 Overview of four cases for membrane reinforcement.

$$\begin{cases} n_{xx} < -|n_{xy}| \\ n_{yy} < \dfrac{n_{xy}^2}{n_{xx}} \end{cases} \quad \text{or} \quad \begin{cases} n_{yy} < -|n_{xy}| \\ n_{xx} < \dfrac{n_{xy}^2}{n_{yy}} \end{cases}$$

$$n_{sx} = 0, \quad n_{sy} = 0 \tag{16.8}$$

$$n_{c1,c2} = \frac{n_{xx}+n_{yy}}{2} \pm \sqrt{\left(\frac{n_{xx}-n_{yy}}{2}\right)^2 + n_{xy}^2}$$

Rebar Design and Check on Concrete Stress

The four cases are summarized in Figure 16.2. The formulas correspond to the optimum direction of concrete compression, i.e., the θ value leading to the minimum amount of reinforcement. The reinforcement design is obtained from

$$a_{sx} = \frac{n_{sx}}{f_{syd}}, \quad a_{sy} = \frac{n_{sy}}{f_{syd}} \tag{16.9}$$

where a_{sx} and a_{sy} are steel areas per unit length and f_{syd} is the design yield strength of the reinforcement. The concrete stress is given by

$$f_c = -n_c / h \tag{16.10}$$

which must be checked against the design compressive strength f_{cd}. For this strength we can apply the Model Code 1990 or the Practitioners guide of *fib*.

Model Code 1990

The Model Code 1990 recommends

$$\begin{aligned} &\text{Case 1 to 3} \quad f_c \leq f_{cd2} \\ &\text{Case 4} \qquad\quad f_c \leq K\ f_{cd1} \end{aligned} \tag{16.11}$$

where

$$\begin{aligned} f_{cd1} &= 0.85 \left[1 - \frac{f_{ck}}{250}\right] f_{cd} \\ f_{cd2} &= 0.60 \left[1 - \frac{f_{ck}}{250}\right] f_{cd} \\ K &= \frac{1 + 3.65\alpha}{(1+\alpha)^2}, \quad \alpha = \frac{\sigma_2}{\sigma_1} \end{aligned} \tag{16.12}$$

Here f_{cd} is the design strength of the concrete, f_{ck} is the characteristic strength of the concrete, and σ_1 and σ_2 are the two principal compressive stresses. These formulas are based on experimental studies on biaxial concrete behaviour of Kupfer.

fib Practitioners' Guide

The practitioners'guide of *fib* recommends following planned changes to the ACI code. The proposed formula for the concrete strength is

$$f_{cd} = 0.85 \beta f_{ck} / \gamma_c \tag{16.13}$$

where the factor 0.85 accounts for the variation between the *in-situ* and cylindrical strengths, β accounts for influence of transverse tensile strain, f_{ck} is the characteristic compression strength, and γ_c is the partial safety factor. The formula for β is

$$\beta = \frac{1}{0.8 + 170\varepsilon_1} \tag{16.14}$$

Herein ε_1 is the major principal strain normal to the direction of the concrete struts. For this strain, the yield strain of the steel reinforcement might be chosen, so $\varepsilon_1 = f_{syd}/E$, where E is the Young modulus of steel.

Remark

At this point, it should be pointed out that the discontinuous variation of concrete compressive strength between Cases 3 and 4, or between Cases 2 and 4 does not seem acceptable. This gains special relevance as the expression for f_{cd2} in Eq. (16.12) corresponds practically to an absolute minimum of cracked reinforced concrete. However, this seems to be the price to pay for a simplified design approach.

16.2 Design of Slabs – Normal Moment Yield Criterion

As in an element in membrane state, dimensioning of slabs and shells from internal forces obtained in a finite element analysis is based on an equilibrium model at ultimate state. While careful consideration of the limited ductility of concrete is important in the dimensioning of membrane elements, such a concern is lower for slabs because such structures are typically under-reinforced. Failure is usually governed by yielding of reinforcement, with the exception of point loads, which may result in brittle punching failures in slabs and in shells without transverse reinforcement.

The stress resultants acting in a slab are the bending moments m_{xx} and m_{yy} and twisting moments m_{xy}. For the derivation of the design equations a set of orthogonal axes is chosen in directions x and y, giving moments per unit length m_{xx}, m_{yy} and m_{xy}, such that $m_{yy} > m_{xx}$. The normal moment yield criterion requires that at any point within the element under consideration the moment capacity m_n^* is greater than the applied normal moment m_n for all values of θ, the orientation of the plane measured in the clockwise direction from the x-axis. The resulting formulas for the design moments are presented in Figure 16.4. Reinforcement is provided in the x- and y-directions to resist design ultimate moments m_{xb}, m_{xt}, m_{yb} and m_{yt}. The subscripts b and t indicate bending moments giving tension in the slab bottom and slab top, respectively. The bottom is at the positive z-side of the slab middle plane, and

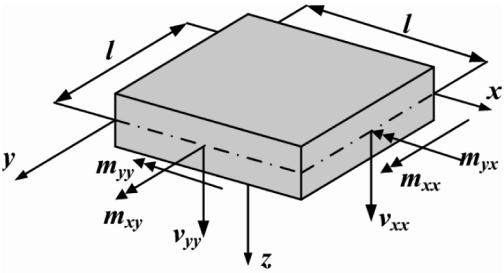

Figure 16.3 Stress resultants acting in a slab.

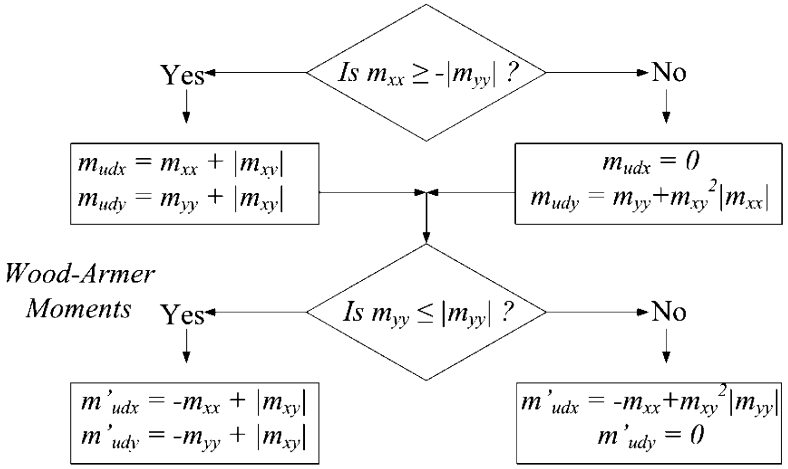

Figure 16.4 Reinforcement for slabs according to ENV 1992-1-1.

the top at the negative side. The shown equations are used in many software packages for slab reinforcement design. Often, only the top-left corner of the figure is used.

Evaluation

The use of the equations of Figure 16.4 is *discouraged* for a number of reasons. The equations are not able to take into consideration transverse shear forces, do not check for concrete crushing, and do not fulfill equilibrium requirements. We strongly recommend using the three layer sandwich model, which applies for slabs and shells. This is the subject of Sections 16.3 and 16.4.

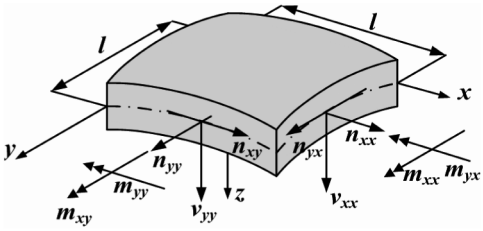

Figure 16.5 Stress resultants acting on a shell element.

16.3 Slab and Shell Elements – Basic Model

The problem to be discussed in this section is the design of a shell element of thickness h, subjected to combined membrane forces and bending moments and where the directions of the principal flexural and membrane forces do not, in general, coincide. A slab element is a special case of the stated problem. Figure 16.5 shows the applied forces and moments. These forces and moments have to be in equilibrium with the tensile forces in the reinforcement and the compressive forces in the concrete. We choose a set of x,y,z-axes as we did in Chapters 3 and 4, where z is pointing downward, see Figure 16.5. The reinforcement consists again of a mesh of orthogonal rebars parallel to the x,y-axes, now placed in a upper and lower layer. We refer to the upper or top layer by the subscript t (negative z-side) and to the lower or bottom layer by b (positive z-side).

The formulation of this problem is identical to that in Section 16.1 for membrane states. Again, the total resistance of the element is obtained by adding the concrete and reinforcement contributions. We model the shell element as a three layer sandwich, shown in Figure 16.6. The outer layers are covers of the sandwich and the inner layer is the core. The cover layers provide resistance to the in-plane effects of flexure and membrane loading, while the core provides a shear transfer between the covers. The thickness of the covers is a and the distance between the middle planes of the covers is d_v.

Dependent on its size, the transverse shear force has an impact on the amount of the reinforcement in the covers. Small values have no impact, large values do, To decide whether the shear forces are small we must consider the maximum shear force v_o as specified in Eq. (4.24).

$$v_o = \sqrt{v_x^2 + v_y^2} \tag{16.15}$$

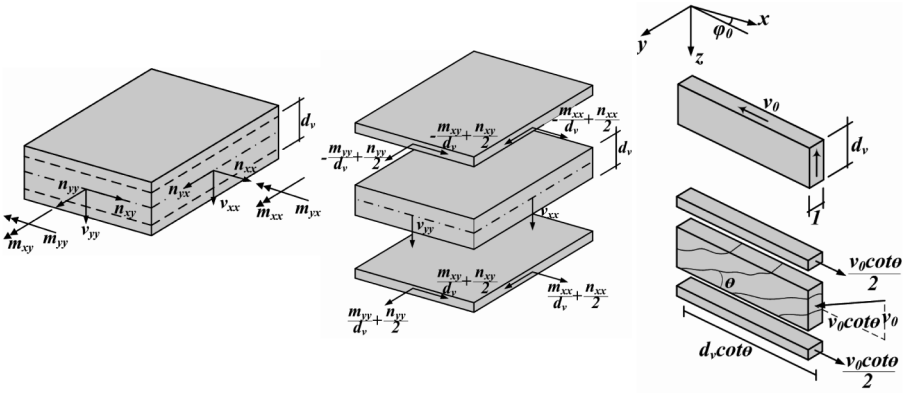

Figure 16.6 Sandwich model: (a) shell element; (b) layer forces; (c) transverse shear transfer in uncracked and cracked concrete, Marti (1991).

which acts in the direction of an angle β_o with the x-axis, defined by Eq. (4.23).

$$\tan \beta_o = \frac{v_y}{v_x} \tag{16.16}$$

This shear force is small if it is below the shear cracking resistance d_v, $\tau_{c,\text{red}}$, where $\tau_{c,\text{red}}$ is the nominal strength of the slab without transverse reinforcement. Then the core will remain uncracked. For the value $\tau_{c,\text{red}}$, we may apply ENV 1992-1-1, which provides

$$\tau_{c,\text{red}} = 0.25 f_{ctd}(1.6 - d_v)(1.2 + 40\rho_l) + 0.15\sigma_{cp} \tag{16.17}$$

Here, f_{ctd} is the design tensile strength, d_v is the internal lever arm in meters, ρ_l is the percentage of longitudinal reinforcement, and σ_{cp} is the in-plane normal compressive stress. If significant tensile membrane forces are applied to the element, $t_{c,\text{red}}$ should be taken to be zero. Provided that no significant tensile membrane forces exist, the expression in Eq. (16.17) can be simplified to a lower bound, neglecting the positive effect of the longitudinal reinforcement.

$$\tau_{c,\text{red}} = 0.30 f_{ctd}(1.6 - d_v) \tag{16.18}$$

16.3.1 Basic Model – No Cracking Due to Transverse Shear

We start with small shear forces. The core layer is supposed not to crack and to be able to carry transverse shear forces. Figure 16.6b depicts the sandwich

model for this case. The need for reinforcement needs to be investigated only for the combination of membrane forces and bending and twisting moments. It is important to decide what thicknesses are assigned to the top and bottom layers. In the basic model these thicknesses are not explicitly considered. It is assumed that all reinforcement layers coincide in each outer sandwich layer, which also coincides with the resultants of the concrete compression forces. Therefore one lever arm d_v applies for both directions x and y. The membrane forces in the external layers are given by

$$\begin{aligned} n_{xxt} &= \frac{m_{xx}}{d_v} + \frac{n_{xx}}{2}, \quad n_{xxb} = -\frac{m_{xx}}{d_v} + \frac{n_{xx}}{2} \\ n_{yyt} &= \frac{m_{yy}}{d_v} + \frac{n_{yy}}{2}, \quad n_{yyb} = -\frac{m_{yy}}{d_v} + \frac{n_{yy}}{2} \\ n_{xyt} &= \frac{m_{xy}}{d_v} + \frac{n_{xy}}{2}, \quad n_{xyb} = -\frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} \end{aligned} \tag{16.19}$$

Using the expressions provided above for the cover membrane elements, we can obtain the final expressions for the forces per unit width for the reinforcement design in shells (assuming that all four reinforcements are needed).

$$\begin{aligned} n_{sxt} &= \frac{m_{xx}}{d_v} + \frac{n_{xx}}{2} + \left| \frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} \right| \\ n_{syt} &= \frac{m_{yy}}{d_v} + \frac{n_{yy}}{2} + \left| \frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} \right| \\ n_{sxt} &= -\frac{m_{xx}}{d_v} + \frac{n_{xx}}{2} + \left| -\frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} \right| \\ n_{syt} &= -\frac{m_{yy}}{d_v} + \frac{n_{yy}}{2} + \left| -\frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} \right| \end{aligned} \tag{16.20}$$

From these forces we can derive reinforcement percentages:

$$\rho_{xt} = \frac{n_{sxt}}{hf_{syd}}, \quad \rho_{yt} = \frac{n_{syt}}{hf_{syd}}, \quad \rho_{xb} = \frac{n_{sxb}}{hf_{syd}}, \quad \rho_{yb} = \frac{n_{syb}}{hf_{syd}} \tag{16.21}$$

where f_{syd} is the design yield stress of steel.

16.3.2 Basic Model – Cracking Due to Transverse Shear

If the transverse shear forces are high enough to produce cracking of the sandwich core, additional reinforcement is required. Here we follow the approach as proposed by Marti [27]. An alternative proposal can be found in

Eurocode 2 [23]. The core is treated like the web of a girder of a flanged cross-section running in the β_o-direction of the maximal shear force. Figure 16.6c shows that concrete struts in the core come into being at an angle θ with the middle plane. To ensure equilibrium, additional membrane forces must occur in the upper and lower cover. Choosing $\theta = 45°$ leads to additional membrane forces in both covers of size v_o in the direction of the maximal shear force. The choice of 45° for the crack angle in the core conforms to the traditional Morsch truss for reinforced concrete beams. Decomposing the additional membrane force in the covers to membrane forces in the x- and y-direction leads to the following expressions:

$$\begin{aligned}
n_{xxt} &= \frac{m_{xx}}{d_v} + \frac{n_{xx}}{2} + \frac{v_x^2}{2v_o}, \quad & n_{xxb} &= -\frac{m_{xx}}{d_v} + \frac{n_{xx}}{2} + \frac{v_x^2}{2v_o} \\
n_{yyt} &= \frac{m_{yy}}{d_v} + \frac{n_{yy}}{2} + \frac{v_y^2}{2v_o}, \quad & n_{yyb} &= -\frac{m_{yy}}{d_v} + \frac{n_{yy}}{2} + \frac{v_y^2}{2v_o} \\
n_{xyt} &= \frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} + \frac{v_x v_y}{2v_o}, \quad & n_{xyb} &= -\frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} + \frac{v_x v_y}{2v_o}
\end{aligned} \tag{16.22}$$

Using these expressions for membrane elements, we find the final expressions for the design of reinforcement for shells (assuming that all four reinforcements are needed):

$$\begin{aligned}
n_{sxt} &= \frac{m_{xx}}{d_v} + \frac{n_{xx}}{2} + \frac{v_x^2}{2v_o} + \left| \frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} + \frac{v_x v_y}{2v_o} \right| \\
n_{syt} &= \frac{m_{yy}}{d_v} + \frac{n_{yy}}{2} + \frac{v_y^2}{2v_o} + \left| \frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} + \frac{v_x v_y}{2v_o} \right| \\
n_{sxt} &= -\frac{m_{xx}}{d_v} + \frac{n_{xx}}{2} + \frac{v_x^2}{2v_o} + \left| -\frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} + \frac{v_x v_y}{2v_o} \right| \\
n_{syt} &= -\frac{m_{yy}}{d_v} + \frac{n_{yy}}{2} + \frac{v_y^2}{2v_o} + \left| -\frac{m_{xy}}{d_v} + \frac{n_{xy}}{2} + \frac{v_x v_y}{2v_o} \right|
\end{aligned} \tag{16.23}$$

From these forces, we derive reinforcement percentages by using the formulas in (16.21). Transverse reinforcement is needed with a percentage ρ_z given by

$$\rho_z = \frac{v_o}{d_v f_{syd}} \tag{16.24}$$

In practical problems we recommend increasing the slab or shell thickness so that transverse reinforcement is avoided.

16.3.3 Evaluation

The basic sandwich model is simple to apply, but definitely is an approximation to reality. We mention the following:

- It is assumed that the core does not contribute to transferring membrane forces. Compatibility requirements show that this cannot be correct.

It is assumed that both reinforcement layers in an outer layer are positioned in the middle plane of the cover; this is physically impossible, and that this middle plane coincides with the resultant of the respective compression force in the concrete, which is not normally the case.

The angle β_o for the cracks in the core due to the transverse shear force has been tacitly assumed to be $\pm 45°$.

Even if global equilibrium is satisfied, several terms associated with different lever arms of all forces are not considered; this means that deviance from reality tends to increase for higher reinforcement percentages or large twisting moments. In these cases, the thickness of the concrete layer tends to increase and the basic model is unsafe.

In Section 16.4 we present an advanced sandwich model. It also starts from the supposition of a three layer sandwich model and division of force transfer, such that the covers carry the membrane forces, bending and twisting moments and the core carries the shear forces. For the rest the shortcomings of the basic model are fully repaired. We assign its own plane to each reinforcement layer, permit the thickness of covers to be adequately calculated and rigorously consider the missing terms. So a consistent set of hypotheses lays the foundation of the advanced model.

16.4 Formulation of the Advanced Three-layer Model

In the consistent model, the internal lever d_v is not assumed *a priori* and it is not equal in all directions, being calculated using an iterative process. Four different cases must be analyzed and treated separately: (a) reinforcement needed in both outer layers; (b) reinforcement needed only in the bottom layer; (c) reinforcement needed only in the top layer; (d) no need for reinforcement. The complete formulation of the problem, the software code and validation can be found in [25, 26]. The described phenomena are simple but the resulting equations are reasonably complex, leading to an indeterminate system of nonlinear equations.

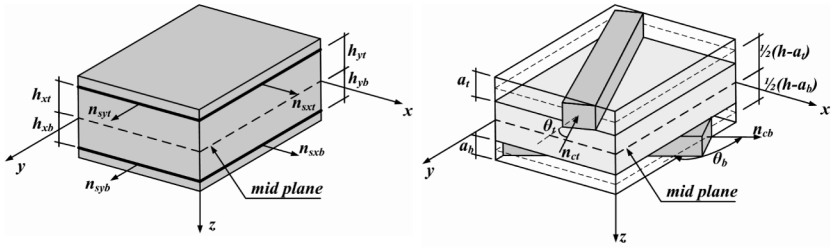

Figure 16.7 Definition of advanced sandwich model.

The geometry of the advanced model is shown in Figure 16.7. We introduce different distances h_{xt}, h_{yt}, h_{xb} and h_{yb} for the four reinforcement layers to the middle plane of the slab. The thicknesses of the outer layers are a_t and a_b, respectively. The core between these layers has thickness h_c. As for the basic model, we define resisting reinforcement forces n_{sxt}, n_{syt}, n_{sxb} and n_{syb}. The two forces for the x-direction are summed to n_{sx} and for the y-direction to n_{sy}. Correspondingly resisting reinforcement moments m_{sx} and m_{sy} are defined. For the concrete top and bottom layer we introduce resisting forces n_{ct} and n_{cb}, respectively, and resisting concrete moments m_{ct} and m_{cb}. Here, subscripts s and c indicate steel and concrete, respectively, and subscripts t and b indicate again top and bottom external layer, respectively.

Case 1: Reinforcement in Both Outer Layers

If reinforcement is needed in the outer layers, the resisting forces and moments for the reinforcement in the x- and y-directions are given by

$$\begin{aligned} n_{sx} &= n_{sxt} + n_{sxb} \\ n_{sy} &= n_{syt} + n_{syb} \end{aligned} \tag{16.25}$$

$$\begin{aligned} m_{sx} &= -n_{sxt}h_{xt} + n_{sxt}h_{xb} \\ m_{sy} &= -n_{syt}h_{yt} + n_{syt}h_{yb} \end{aligned} \tag{16.26}$$

and for the concrete by

$$\begin{aligned} n_{ct} &= -a_t f_c \\ n_{cb} &= -a_b f_c \end{aligned} \tag{16.27}$$

$$m_{ct} = -\frac{1}{2}(h - a_t)\, n_{ct}$$
$$m_{cb} = \frac{1}{2}(h - a_b)\, n_{cb} \tag{16.28}$$

Equations (16.25)–(16.27) provide the internal forces and moments. Equilibrium with the applied set of forces and moments leads to

$$\begin{aligned} n_{xx} &= n_{sx} + n_{ct}\sin^2\theta_t + n_{cb}\sin^2\theta_b \\ n_{yy} &= n_{sy} + n_{ct}\cos^2\theta_t + n_{cb}\cos^2\theta_b \\ n_{xy} &= -n_{ct}\sin\theta_t\cos\theta_t - n_{cb}\sin\theta_b\cos\theta_b \end{aligned} \tag{16.29}$$

$$\begin{aligned} m_{xx} &= m_{sx} + m_{ct}\sin^2\theta_t + m_{cb}\sin^2\theta_b \\ m_{yy} &= m_{sy} + m_{ct}\cos^2\theta_t + m_{cb}\cos^2\theta_b \\ m_{xy} &= -m_{ct}\sin\theta_t\cos\theta_t - m_{cb}\sin\theta_b\cos\theta_b \end{aligned} \tag{16.30}$$

Equation (16.29) correspond to the membrane forces, while Eq. (16.32) correspond to bending equations. If $\theta_t \neq 0, \pi/2$ and $\theta_b \neq 0, \pi/2$, the previous equations can be recast as

$$-n_{ct} = \frac{(h - a_t)n_{xy} - 2m_{xy}}{h_c \sin 2\theta_t}$$
$$-n_{cb} = \frac{(h - a_b)n_{xy} + 2m_{xy}}{h_c \sin 2\theta_b} \tag{16.31}$$

Reinforcement will be given upon solving Eqs. (16.25) to (16.30). The objective is to calculate the forces in the reinforcement n_{sxt}, n_{syt}, n_{sxb} and n_{syb}. The other unknowns are a_t, a_b, θ_t and θ_b. Therefore the system of six equations contains eight unknowns. This means that the values of θ_t and θ_b should be chosen so that the total amount of reinforcement is minimized. The values of $\theta_t = \theta_b = \pi/4$ and $a_t = a_b = 0.2h$ can be assumed as an initial guess. Setting the values of θ to $\pi/4$ minimizes the total reinforcement in membrane elements. Setting $a = 0.2h$ is usual for beam sections. The values are then adjusted by an iterative procedure until equilibrium is fulfilled. The reader is referred to [25,26] for a full description of the iterative method.

Compressive crushing is checked by enforcing that $a_t + a_b \leq h$ and tensile reinforcement is calculated assuming yielding of the reinforcement.

Case 2: Reinforcement in Bottom Layer Only

For biaxial compression in the top layer, reinforcement in the top layer is not needed. We indicate the concrete top layer membrane forces by n_{cxxt}, n_{cyyt} and n_{cxyt}. The forces and moments that the reinforcement resists in the x, y-directions are given by

$$\begin{aligned} n_{sx} &= n_{sxb} \\ n_{sy} &= n_{syb} \\ m_{sx} &= n_{sxb} h_{xb} \\ m_{sy} &= n_{syb} h_{yb} \end{aligned} \tag{16.32}$$

and by the concrete bottom layer are

$$\begin{aligned} n_{cb} &= -a_b f_c \\ m_{cb} &= \frac{1}{2}(h - a_b) n_{cb} \end{aligned} \tag{16.33}$$

Equilibrium with the applied set of forces and moments yields

$$\begin{aligned} n_{xx} &= n_{sx} + n_{cxt} + n_{cb} \sin^2 \theta_b \\ n_{yy} &= n_{sy} + n_{cyt} + n_{cb} \cos^2 \theta_b \\ n_{xy} &= n_{cxyt} - n_{cb} \sin \theta_b \cos \theta_b \end{aligned} \tag{16.34}$$

$$\begin{aligned} m_{xx} &= m_{sx} + m_{cxt} + m_{cb} \sin^2 \theta_b \\ m_{yy} &= m_{sy} + m_{cyt} + m_{cb} \cos^2 \theta_b \\ m_{xy} &= m_{cxyt} - m_{cb} \sin \theta_b \cos \theta_b \end{aligned} \tag{16.35}$$

with

$$\begin{aligned} m_{cxt} &= -\frac{1}{2}(h - a_t)\, n_{cxt} \\ m_{cyt} &= -\frac{1}{2}(h - a_t)\, n_{cyt} \\ m_{cxyt} &= -\frac{1}{2}(h - a_t)\, n_{cxyt} \end{aligned} \tag{16.36}$$

In the current case there are still eight unknowns. However, one extra equation must be added to the six equations of equilibrium, representing the biaxial state of stress in the concrete top layer

$$n_{ct} = -a_t f_c = \frac{n_{cxt} + n_{cyt}}{2} - \sqrt{\left(\frac{n_{cxt} + n_{cyt}}{2}\right)^2 + n_{cxyt}^2} \tag{16.37}$$

Here, f_c has a higher value than the uni-axial compressive strength of cylinders due to biaxial confinement. Nevertheless there are eight unknowns and seven equations, meaning that θ_b should be chosen so that the total amount of reinforcement is minimized.

Case 3: Reinforcement in Top Layer Only

The case of biaxial compression in the bottom layer is identical to the case of biaxial compression in the top layer, with a rotation of indices. Therefore establishing the equilibrium equations requires no additional explanation.

Case 4: No Reinforcement at All

Finally, in the case of biaxial compression in top and bottom layers, there is no need of reinforcement and the solution is unique. Assuming that the concrete top layer membrane forces are $n_{cxt,ncyt}$ and n_{cxyt} respectively in the x, y-direction and as shear force, and the concrete bottom layer membrane forces are n_{cxb}, n_{cyb} and n_{cxyb} with a similar meaning, the equilibrium equations might be written as

$$\begin{aligned} n_{xx} &= n_{cxt} + n_{cxb} \\ n_{yy} &= n_{cyt} + n_{cyb} \\ n_{xy} &= n_{cxyt} + n_{cxyb} \end{aligned} \tag{16.38}$$

$$\begin{aligned} m_{xx} &= m_{cxt} + m_{cxb} \\ m_{yy} &= m_{cyt} + m_{cyb} \\ m_{xy} &= m_{cxyt} + m_{cxyb} \end{aligned} \tag{16.39}$$

with

$$\begin{aligned} m_{cxt} &= -\frac{1}{2}(h - a_t)n_{cxt} \quad m_{cxb} = \frac{1}{2}(h - a_b)n_{cxb} \\ m_{cyt} &= -\frac{1}{2}(h - a_t)n_{cyt} \quad m_{cyb} = \frac{1}{2}(h - a_b)n_{cyb} \\ m_{cxyt} &= -\frac{1}{2}(h - a_t)n_{cxyt} \quad m_{cxyt} = \frac{1}{2}(h - a_b)n_{cxyt} \end{aligned} \tag{16.40}$$

The principal concrete compression forces in each layer may be calculated according to

$$\begin{aligned} n_{c,t} &= \frac{n_{cxt} + n_{cyt}}{2} \pm \sqrt{\left(\frac{n_{cxt} - n_{cyt}}{2}\right)^2 + n_{cxyt}^2} \\ n_{c,b} &= \frac{n_{cxb} + n_{cyb}}{2} \pm \sqrt{\left(\frac{n_{cxb} - n_{cyb}}{2}\right)^2 + n_{cxyb}^2} \end{aligned} \tag{16.41}$$

and the layer thickness may be calculated according to the MC90 as

$$a_t = -\frac{n_{ct,\max}}{K f_{cd1}}, \quad a_b = -\frac{n_{cb,\max}}{K f_{cd1}} \tag{16.42}$$

As shown above, there are eight unknowns and eight equations (the six equilibrium equations and two equations to check the maximum compressive stress in the layers), meaning that the problem is determinate.

16.5 Applications on Element Level

In this section we illustrate the use of the basic and advanced model for two elements. The first one is subjected to a combination of a membrane force and bending moment. The second is a slab element subjected to a twisting moment.

16.5.1 Element with Membrane Force and Bending Moment

An element is subjected to an applied set of a bending moment and membrane shear force given by $m_{xx} = 235$ kNm/m and $n_{xy} = 1{,}806$ kN/m. The material properties of concrete and steel are $f_c = 41.8$ MPa and $f_{sy} = 492$ MPa. The location of the reinforcement is given by $h_{xt} = h_{xb} = 0.122$ m and $h_{yt} = h_{yb} = 0.100$ m. This element is chosen because an experimental result of Kirsher and Collins is available for the purpose of comparison [28]. The bending moment and membrane shear force mentioned above are the ultimate loads in the test. We will start from this load and determine how much reinforcement is needed.

The top row in Table 16.1 shows that in total 111.4 cm^2/m reinforcement is applied in the element in the test. Not all the reinforcement yielded at

Table 16.1 Reinforcement for membrane shear + bending moment.

	Reinforcement areas (cm^2/m)				
Method	x upper	y upper	x lower	y lower	Total
Experiment	41.8	13.9	41.8	13.9	111.4
Nonlinear	0.0	14.1	37.6	16.9	68.6
Basic method	0.0	15.7	39.9	18.4	74.0
Advanced method	0.0	16.6	36.8	17.9	71.3

failure. The second row in the table is the prediction of the needed reinforcement on the basis of a nonlinear analysis by an iterative computer program with optimization [29]. This provided a minimum amount of reinforcement equal to 68.6 cm^2/m. The third row presents the results of the basic sandwich model and the fourth row of the advanced model. So we use the two rows 'Experiment' and 'Nonlinear' are benchmark references for the basic and advanced method.

Basic Model

For the basic model an average distance of layer centres to the middle plane of the element of 0.111 m is chosen. Therefore $d_v = 0.222$ m. On the basis of Eq. (16.19) we find the following

$$
\begin{aligned}
n_{xxb} &= 235/0.222 = 1{,}059 \text{ kN/m},\\
n_{xxt} &= -235/0.222 = -1{,}059 \text{ kN/m},\\
n_{yyb} &= 0,\\
n_{yyt} &= 0,\\
n_{xyb} &= 1{,}806/2 = 903 \text{ kN/m},\\
n_{xyt} &= 1{,}806/2 = 903 \text{ kN/m}.
\end{aligned}
$$

Using the expressions for membrane elements it is possible to obtain, for the top layer (x reinforcement not needed)

$$
\begin{aligned}
n_{sxt} &= 0,\\
n_{syt} &= n_{yyt} - \frac{n_{xyt}^2}{n_{xxt}} = 0 - \frac{903^2}{-1{,}059} = 770 \text{ kN/m}
\end{aligned}
$$

$$n_{ct} = -1{,}059 + \frac{903^2}{-1{,}059} = -1{,}829 \text{ kN/m}$$

$$a_{sxt} = 0$$

$$a_{syt} = \frac{n_{syt}}{f_{sy}} = \frac{770}{492 \times 10^3} \times 10^4 = 15.7 \text{ cm}^2/\text{m}$$

$$a_t = -\frac{n_{ct}}{f_c^{\text{eff}}} - \frac{-1{,}829}{0.6 \times 41.8 \times 10^3} = 0.073 \text{ m}$$

Note that the value of the effective compressive strength was here assumed as $0.6 f_c$. Similarly, for the bottom layer

$$n_{sxb} = n_{xxb} + \left|n_{xyb}\right| = 1{,}059 + 903 = 1{,}962 \text{ kN/m}$$

$$n_{syb} = n_{xyb} + \left|n_{xyb}\right| = 0 + 903 = 903 \text{ kN/m}$$

$$n_{cb} = -2\left|n_{xyb}\right| = -2 \times 903 = 1{,}806 \text{ kN/m}$$

$$a_{sxb} = \frac{n_{sxb}}{f_{sy}} = \frac{1{,}962}{492 \times 10^3} \times 10^4 = 39.9 \text{ cm}^2/\text{m}$$

$$a_{sxt} = \frac{n_{sxt}}{f_{sy}} = \frac{903}{492 \times 10^3} \times 10^4 = 18.4 \text{ cm}^2/\text{m}$$

$$a_b = -\frac{-1{,}806}{0.6 \times 41.8 \times 10^3} = 0.072 \text{ m}$$

Advanced Model

The advanced sandwich model requires five iterations, provides the thickness of the layers equal to 0.072 m and 0.075 m for the top and bottom respectively, and the reinforcement results as given in the third row of Table 16.1. It can be seen that the results are almost the same as the basic sandwich model. If the results of the nonlinear analysis are assumed as reference values, the basic sandwich model provides +8% and the advanced sandwich model provides +4% of the total reinforcement.

16.5.2 Slab Element with Twisting Moment

A slab element is subjected to pure torsion by an applied twisting moment. The value of the twisting moment is one time chosen $m_{xy} = 42.5$ kNm/m

Table 16.2 Reinforcement for small twisting moment.

	Reinforcement areas (cm^2/m)				
Method	x upper	y upper	x lower	y lower	Total
Experiment	5.0	5.0	5.0	5.0	20.0
Normal moment	5.0	5.3	5.0	5.3	20.6
Basic model	5.6	5.6	5.6	5.6	23.4
Advanced model	5.0	5.0	5.0	5.0	20.0

and one time $m_{xy} = 101.5$ kNm/m. These values are chosen because results of a test. Marti, Leesti and Khalifa obtained them for a lightly reinforced (0.25%) and a severely reinforced (1.0%) element, respectively [30]. We will use these moments as load and determine the needed reinforcement. The material properties are $f_c = 44.4$ MPa, and $f_{sy} = 479$ MPa for the light reinforcement and $f_{sy} = 412$ MPa for the severe reinforcement. The location of the reinforcement is given by $h_{xt} = h_{xb} = 0.073$ m and $h_{yt} = h_{yb} = 0.084$ m, for the light one, and $h_{xt} = h_{xb} = 0.066$ m and $h_{yt} = h_{yb} = 0.082$ m, for the severe one.

Table 16.2 regards the small twisting moment. The line 'Experiment' lists the reinforcement existing in the tested element, which is the same in x- and y-direction, and in the top and bottom layer (5 cm^2/m). The second line in the table regards the normal moment method. This violates equilibrium as different reinforcements are calculated for each direction. This is in agreement with the formulation, as different lever arms are found for each reinforcement direction, but equilibrium requires the forces in all reinforcements to be the same. The basic and advanced sandwich models fulfill equilibrium correctly. The normal moment method provides a reasonable (conservative) value of reinforcement for the small twisting moment. The reinforcement is in x-direction about 27% less than the required value and 19% less in y-direction. The basic model is very safe for the small moment. The prediction by the advanced sandwich model is exact for the small moment.

Table 16.3 shows the result for the large twisting moment. In the experiment 20 mm^2/m was used for all four reinforcements. The normal moment method leads to different amounts in x- and y-direction, underestimates the reinforcement in x-direction by 27% and the total by 24%. The basic sandwich model leads to equal amounts, however 19% too small. The advanced sandwich model performs very satisfactory with only 3% difference with the amounts in the test.

Table 16.3 Reinforcement for large twisting moment.

	Reinforcement areas (cm^2/m)				
Method	x upper	y upper	x lower	y lower	Total
Experiment	20.0	20.0	20.0	20.0	80.0
Normal moment	14.6	16.3	14.6	16.3	61.8
Basic model	16.2	16.2	16.2	16.2	64.8
Advanced model	19.4	19.4	19.4	19.4	77.6

The reason for the bad predictions is that the location of the resultant for the forces in the concrete are incorrectly calculated and the interaction between the different forces in reinforcement and concrete are neglected. Therefore, the equations on basis of the normal moment yield criterion and the basic sandwich model should be used with much precaution, or not used at all. Obtained results from these models must be distrusted if high reinforcement ratios are obtained.

16.6 Applications on Structural Level

16.6.1 Deep Beam

We study the one-span deep beam with the geometry and loading illustrated in Figure 16.8(a), borrowed from the special report 'Toward a Consistent Design of Structural Concrete' of the *PCI Journal* [31]. The 0.4 m thick beam carries the axial load of two columns, with 0.5 by 0.4 m^2 cross-section. The span exhibits an opening that modifies the normal deep beam behaviour. The loads specified in Figure 16.8(a) are design loads and the materials employed are concrete class C25/30 ($f_{cd} = 16.67$ MPa) and the steel S500 ($f_{syd} = 435$ MPa).

We discretized the deep beam into eight-noded quadrilateral elements with the mesh schematically illustrated in Figure 16.8(b). We performed the linear-elastic FEM analysis of the deep beam subjected to the design loads, and obtained the in-plane forces n_{xx}, n_{yy} and n_{xy} in each element sampling point (2×2 in-plane integration). Defining the horizontal (x) and the vertical (y) directions as the two orthogonal reinforcement directions, we apply the design giving the amount of steel necessary to resist the calculated forces at each sampling point. Figure 16.9 illustrates the reinforcement needs in

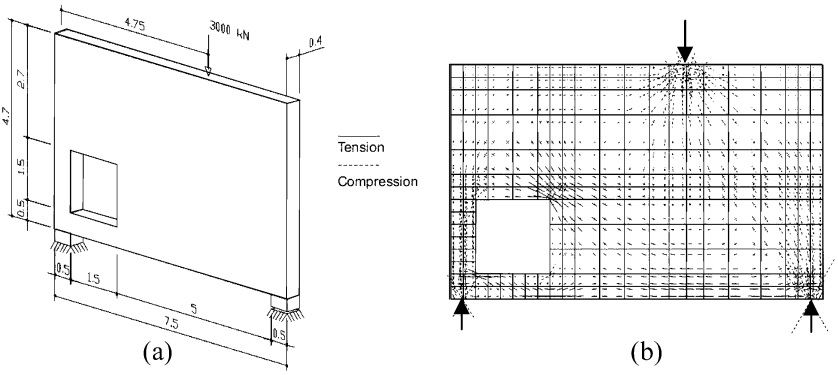

Figure 16.8 Deep beam: (a) geometry and loading (structure from *PCI Journal*, Vol. 32, No. 3); (b) mesh discretization, tension and compression forces and trajectories.

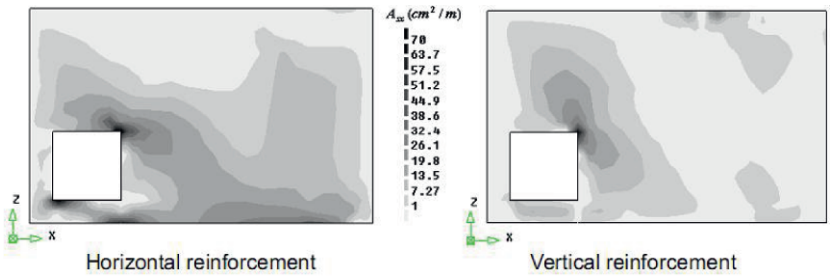

Figure 16.9 Deep Beam: reinforcement needs in the x- and y-directions.

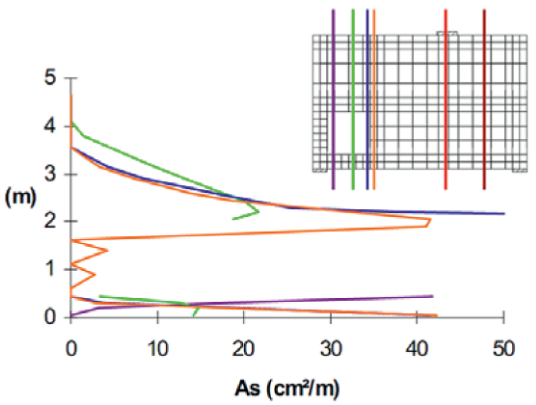

Figure 16.10 Deep beam. Diagrams of horizontal reinforcement at the critical sections.

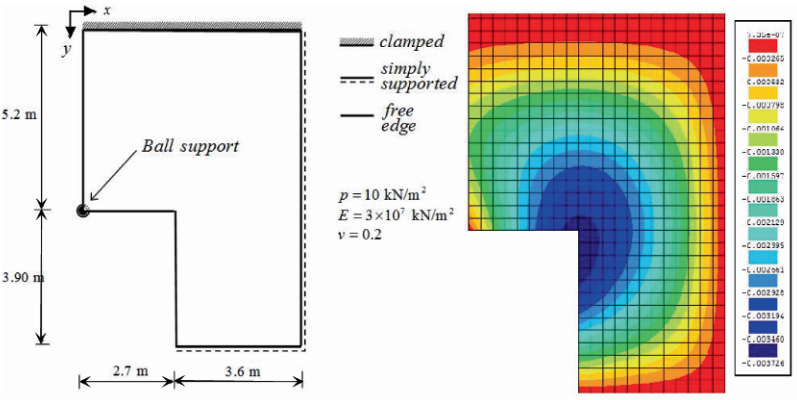

Figure 16.11 Slab: (a) geometry and loading; (b) vertical displacement (m).

the x- and y-directions, respectively. We can easily define the amount and distribution of the reinforcement to be employed by tracing out diagrams of reinforcement at the critical sections as illustrated in Figure 16.10 for the vertical and horizontal reinforcement.

16.6.2 Slab

The L slab, studied in Chapter 12, with the geometry and loading presented in Figure 16.11(a) is now analyzed. This slab has the support conditions

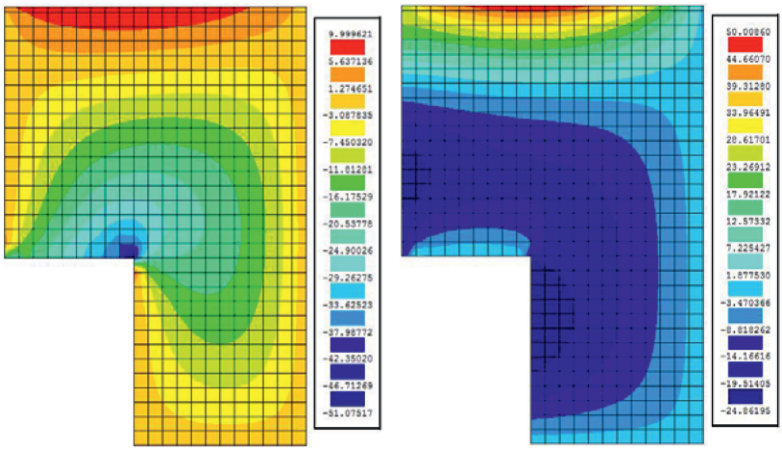

Figure 16.12 Bending moments in the x- and y-directions (kNm/m), respectively.

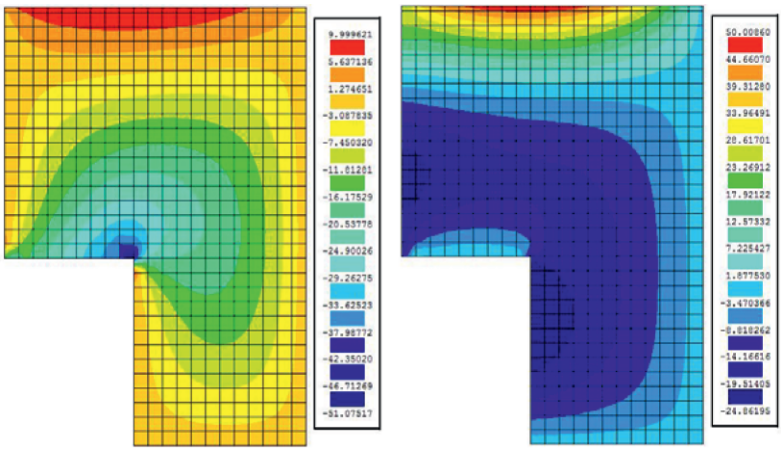

Figure 16.13 Twisting moment (kNm/m) and shear force in x-direction (kN/m).

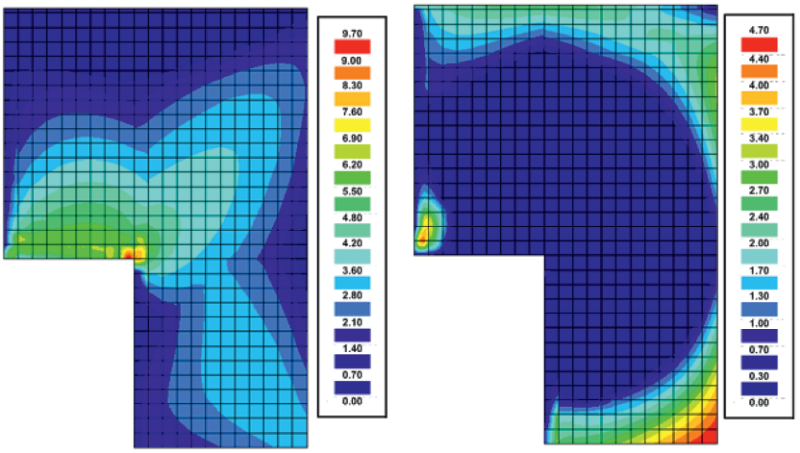

Figure 16.14 Bottom and top reinforcement needs in the x-direction, respectively (cm^2/m).

presented in the Figure 16.11(a) and has a thickness of 0.24 m. The load presented is the design load and the materials are concrete class C25/30 ($f_{cd} = 16.67$ MPa) and the steel S500 ($f_{syd} = 435$ MPa).

We discretize this slab into eight-noded quadrilateral elements with the mesh presented in Figure 16.11(b). We performed the linear-elastic FEM analysis of the slab, and obtained m_{xx}, m_{yy}, m_{xy}, v_x and v_y for all the 2×2 integration points. We can see in Figure 16.12 the bending moments in the x- and y-directions. We illustrate the twisting moment and the shear force in the

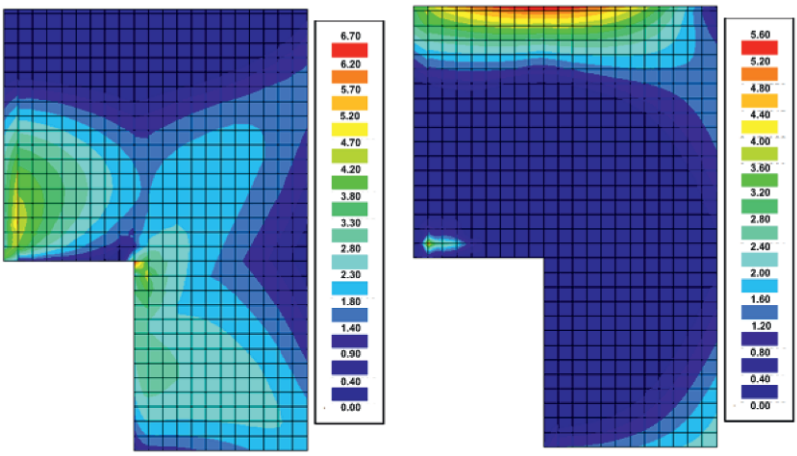

Figure 16.15 Bottom and top reinforcement needs in the y-direction, respectively (cm^2/m).

x-direction in Figure 16.13. Defining the x- and the y-directions as the two orthogonal reinforcement directions, we applied the design model giving the amount of steel necessary to resist the calculated forces at each integration point. Figure 16.14 shows the required reinforcement.

16.7 Message of the Chapter

- Reinforcement in a membrane state (wall) can be designed including the effect of shear forces. Four different cases must be considered, ranging from the need to apply reinforcement in two directions to no reinforcement at all.
- Design of reinforcement in a slab on the basis of the normal moment yield criterion leads to simple, easy to apply formulas. However, we cannot check on concrete crushing, and equilibrium is not satisfied.
- The three layer sandwich model for slabs and shell elements leads to design of reinforcement, a check on concrete crushing, and includes a reinforcement design method for transverse shear forces.

- Two variants of the sandwich model are known, basic and advanced. The basic model is easy to apply. However, it is an unsafe approximation when large twisting moments occur or high percentages of reinforcement are found. The advanced model consistently accounts for the real geometry of the element and yields adequate results.