Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Exam CIE4180: Plate Analysis
Wednesday 10 April 2019, 13:30 – 16:30

This exam consists of 6 questions.
The respective weights are 1.5, 1.5, 2.0, 2.0, 1.5 and 1.5.

Use **separate sheets of paper for each question**!

Write your name and student number **on every page of your work.**

In this exam the notation and sign conventions of the reader (part 1 and 2) and the book (Plates and FEM) have been adopted. **You are supposed to use the same notation and sign convention.**

Question 1

Use **separate sheets of paper for each question**!

Consider the plane stress (or plate membrane) problem of a plate ABCDEGHIJKLMPQR with outer dimensions $2h+r$ by $3h+r$, see the Figure 1.1. The plate has an uniform thickness t. The Young's modulus is E and the Poisson's ratio is ν.

The plate is supported in y-direction along line AB. The plate is supported in x-direction in point A. The plate is loaded by a concentrated in-plane force F at the top of the plate. There is no dead load.

Length h = 416 mm, radius r = 480 mm, thickness t = 10.5 mm, Young's modulus E = 202500 MPa, Poisson ratio ν = 0.31, Force F = 22000 N

In a finite element model of this plate 4-noded (i.e. linear) plane stress elements are used, with an approximate element size of 64 mm.

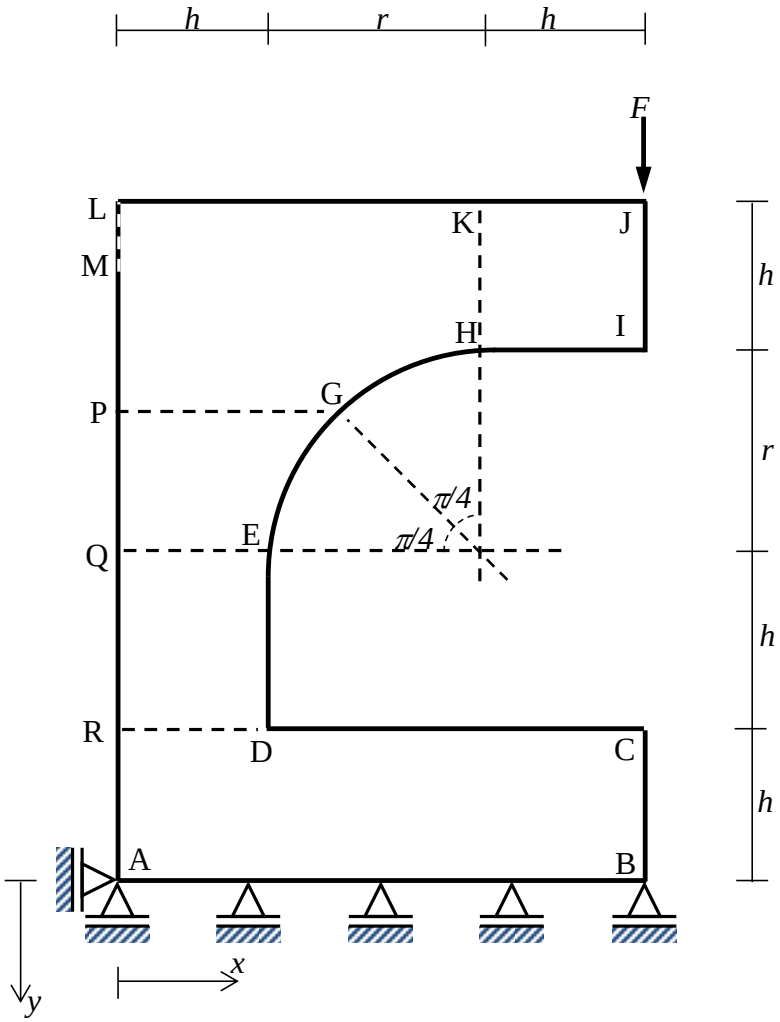


Figure 1.1: plane stress (or plate membrane) problem

We now verify the stress results of the finite element analysis after nodal averaging in a qualitative sense:

a) Rank the resulting σ_{yy} in the points R, D, Q and E, from smallest (the most negative) to largest. Explain shortly.

b) Rank the resulting σ_{xx} in the points Q, E, P and G, from smallest (the most negative) to largest. Explain shortly.

Question 2

Use **separate sheets of paper for each question**!

Consider the same plane stress problem and finite element model as in Question 1. In Figure 2.1 two cross-sections are indicated:

- the straight cross-section KH and
- the zig-zag cross-section LMNOG.

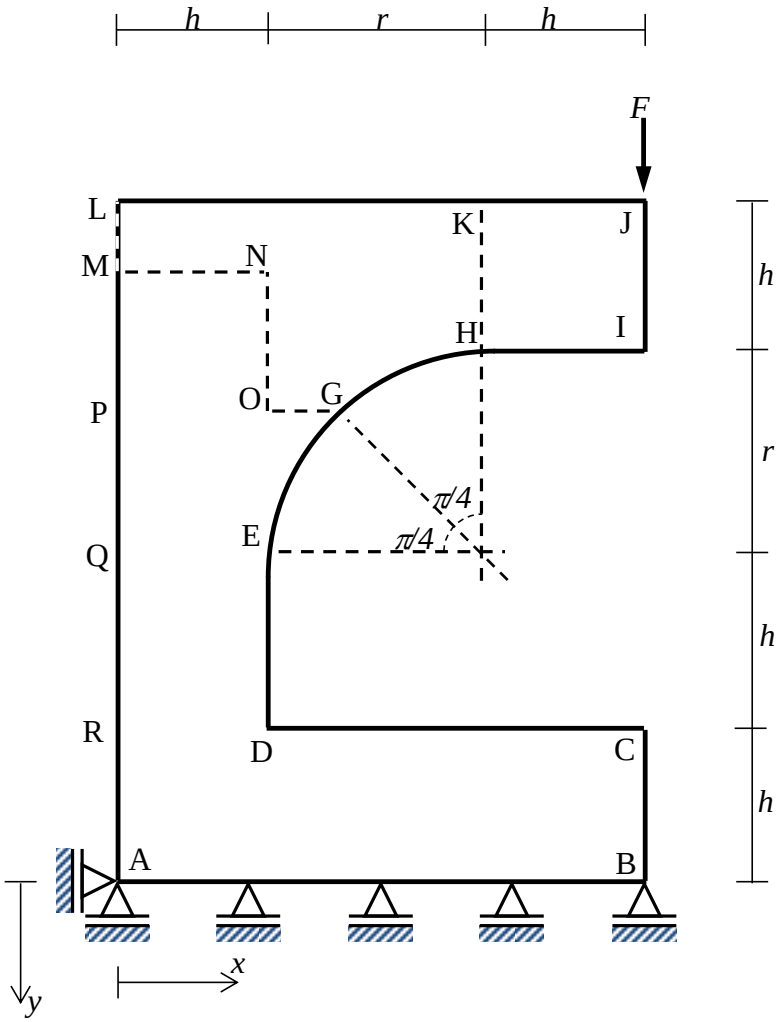


Figure 2.1: plane stress (or plate membrane) problem

You are asked to provide equations for equilibrium verifications along these cross-sections. Write the expression in these equations in an unequivocal way using a notation like $\int_{?_K}^{?_H} \sigma_{??}\, d?$, where the ?'s should be replaced by x or y and where, e.g., x_K denotes the x-coordinate of point K.

a) Provide an equation for the verification of vertical equilibrium using stress components along cross-section KH.

b) Provide an equation for the verification of horizontal equilibrium using stress components along cross-section KH.

c) Provide an equation for the verification of moment equilibrium using stress components along cross-section KH.

d) Provide an equation for the verification of vertical equilibrium using stress components along cross-section LMNOG.

Question 3

Use **separate sheets of paper** **for each question**!

a) What is the unit of m_{xx} ? Choose A, B, C or D.

A Nm
B Nm/m
C Pa
D N/m^2

b) What is the definition of ρ_{xy} for thin plates? Choose A, B, C or D.

A $\rho_{xy} = \dfrac{\partial^2 w}{\partial x \partial y}$

B $\rho_{xy} = -2\dfrac{\partial^2 w}{\partial x \partial y}$

C $\rho_{xy} = \kappa_{xy} + \kappa_{yx}$

D $\rho_{xy} = \dfrac{\partial \kappa_{xy}}{\partial y} - \dfrac{\partial \kappa_{yx}}{\partial x}$

c) Consider the following list. Which equations do not fit in this list? Choose A, B, C or D.

A equilibrium equations
B differential equations
C constitutive equations
D kinematic equations

d) Consider a simply supported plate edge. Which is the boundary condition? Choose A, B, C or D.

A $w = 0,\ \varphi_y = 0$

B $w = 0,\ m_{yy} = 0$

C $m_{yy} = 0,\ v_y + \dfrac{\partial m_{xy}}{\partial x} = 0$

D $m_{yy} = 0,\ \varphi_y = 0$

e) Consider an interior molecule of the finite difference method of a slender beam ($EI\dfrac{d^4 w}{dx^4} = p$).

What is the number of degrees of freedom in this molecule?
What is the number of discrete elements in this molecule?

f) Suppose that you need to model a thin plate but only thick plate elements are available. What action may be practical?
Choose from A, B, C and D. You can choose more than one answer.

A Use very small elements and wait for the computation to finish.
B Spend time on making a special mesh with small elements in the edges.
C Complain to your software company and wait for the new software version.
D Accept unconverged results and convince others that the difference with their results is not important.

g) Who was Gustav Kirchhoff ? Choose A, B, C or D.

A Somebody who gave money to science and in particular to the development of the thin plate theory.
B The German engineer who first applied the thin plate theory to a reinforced concrete bridge.
C A physicist who derived the boundary conditions for the thin plate theory.
D The scientist who proposed the needle hypothesis of the thin plate theory.

Question 4

Use **separate sheets of paper** **for each question**!

A plate is loaded perpendicularly to its plane. Two edges are simply supported, one is fixed and the others are free (Fig. 6). The plate length is 15 m and the width is 10 m. The plate thickness is 190 mm. A distributed load of 10 kN/m^2 is applied in the z-direction. This distributed load includes self-weight.
A linear elastic finite element model is made. The following material properties are used, E = 31500 N/mm^2, ν = 0.2. Four-node plate elements are used (Kirchhoff theory). The element size is 400 mm. The results are shown in Figures 4.1 to 4.6.

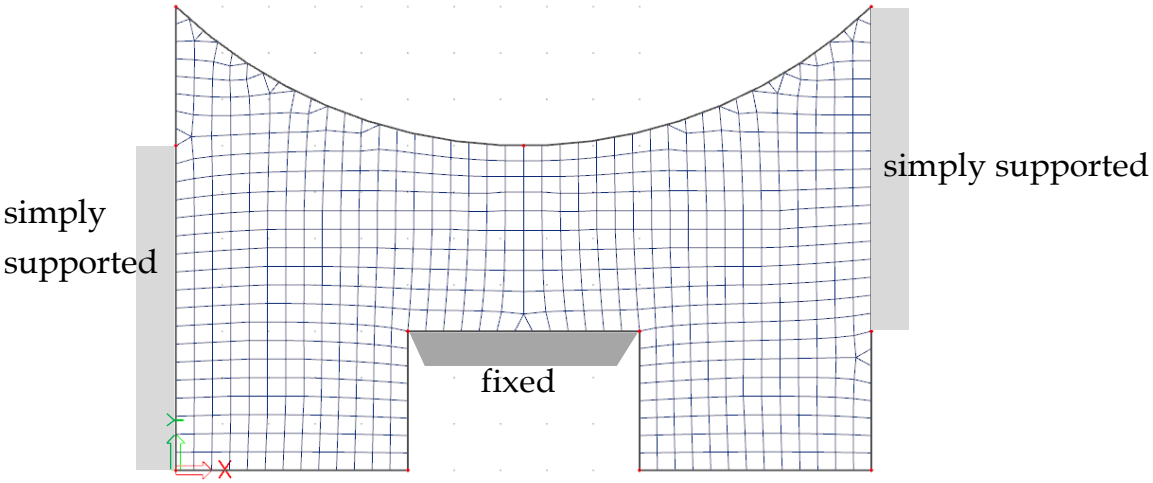


Figure 4.1. Plate with supports and finite element mesh

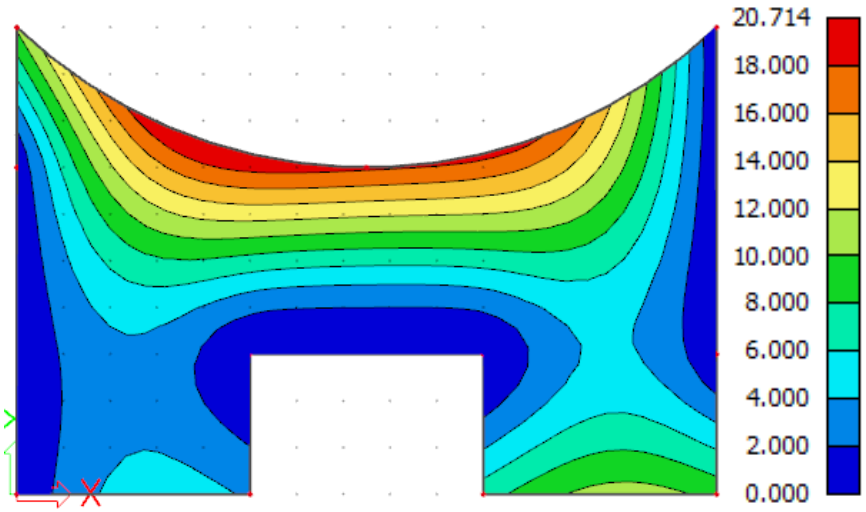


Figure 4.2. Contour plot of the plate deformation [mm]

a) Is this plate thin? Explain your answer.

b) If smaller elements were used, would the computed deflection increase, decrease or stay the same?

c) Does the deformation fulfil the boundary conditions? Explain your answer.

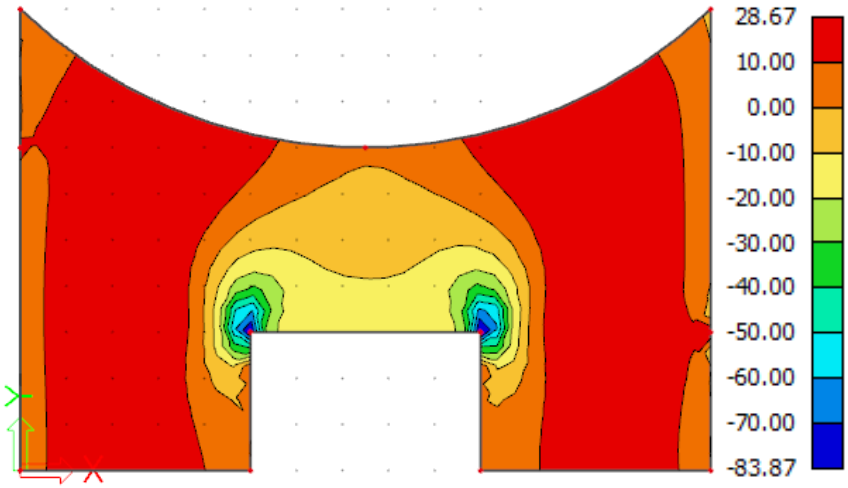


Figure 4.3. Contour plot of the bending moment m_{xx} [kNm/m]

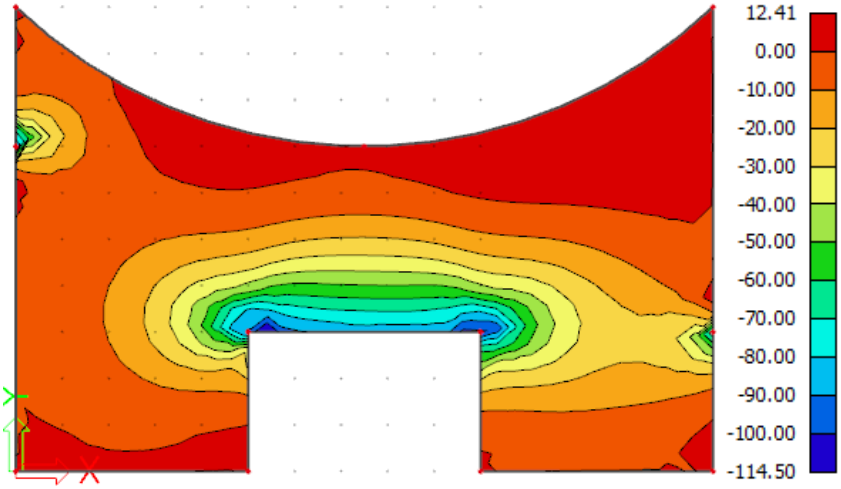


Figure 4.4. Contour plot of the bending moment m_{yy} [kNm/m]

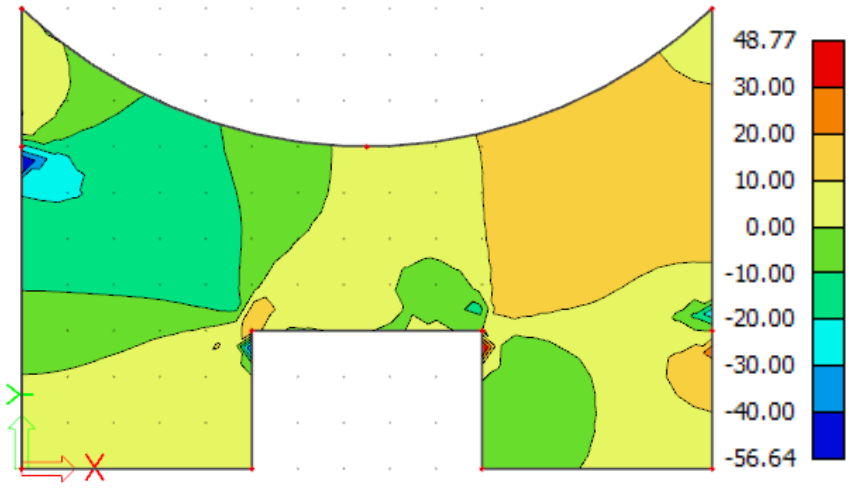


Figure 4.5. Contour plot of the torsion moment m_{xy} [kNm/m]

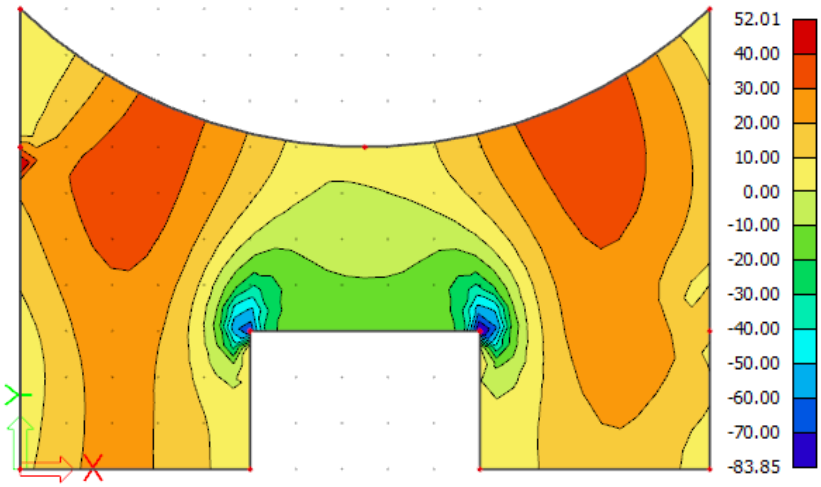


Figure 4.6. Contour plot of the principal moment m_1 [kNm/m]

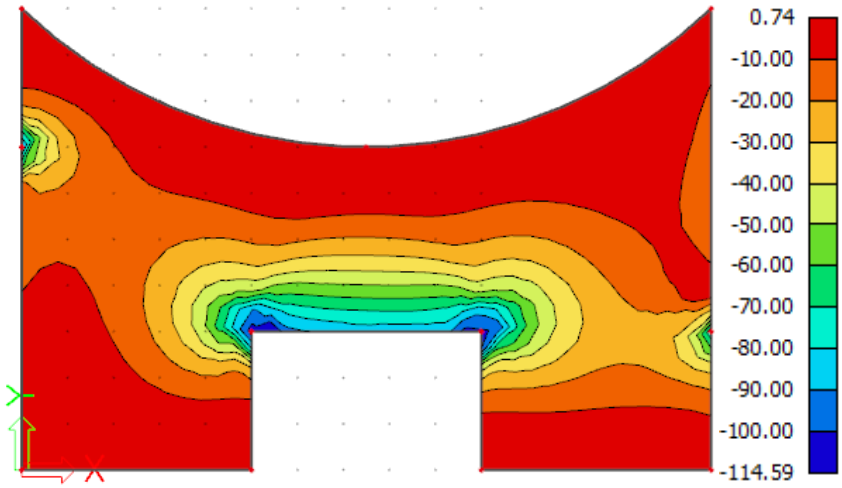


Figure 4.6. Contour plot of the principal moment m_2 [kNm/m]

d) What value is the most negative moment in the plate?
Where does it occur?
We put reinforcing bars in the plate to carry this most negative moment.
In which direction are these bars most efficient?

e) The contour plot of m_{xy} does not show red and it does not show blue.
Why does the software not display these colours?
What do we need to do in order to display red and blue too?

f) Do the moments agree with the boundary conditions? Explain your answer.

g) What is the largest value of the concentrated shear force V in the curved edge?

h) Suppose we want to use the rain shower analogy. What would we need to plot?

Question 5

Use **separate sheets of paper** **for each question**!

A reinforced concrete wall has a thickness of 210 mm. It is loaded by two load combinations. The membrane forces at a particular location are shown in the table below.

load combination 1	load combination 2
n_{xx} = 4300 kN/m n_{yy} = 620 kN/m n_{xy} = -1000 kN/m	n_{xx} = 440 kN/m n_{yy} = -1180 kN/m n_{xy} = 860 kN/m

The reinforcement consists of two meshes. As usual, one mesh is close to the front surface the other mesh is close to the back surface. Each mesh has bars in the *x* direction and bars in the *y* direction. All bars have a diameter of 16 mm and are spaced 180 mm. The yield strength of the reinforcing steel is 370 N/mm^2 (partial factor included). The membrane force capacity due to the bars in the *x* direction is 827 kN/m. The membrane force capacity due to the bars in the *y* direction is also 827 kN/m. The compressive strength of the concrete is 30 N/mm^2 (partial factor included).

a) Suppose that the plate is linear elastic. What are the normal stresses and shear stress?

b) Is the reinforcement sufficient? Explain your answer.

c) Suppose that we want to direct the reinforcement in the principal directions. Over which angle do we need to rotate the reinforcing bars?

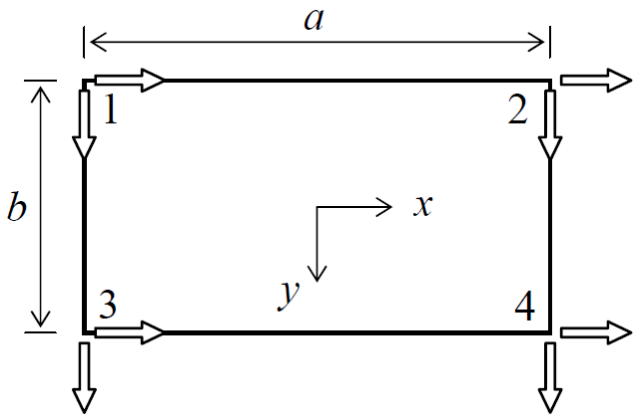
d) Calculate the reinforcement capacities that that are minimally required in the *x* and *y* directions.

e) Suppose that the amount of minimally required reinforcement is larger than the amount of sufficient reinforcement. Is this possible? Explain your answer.

Useful equations
nsx = nxx+|nxy| nsy = nyy+|nxy|
n1 = (nxx+nyy)/2+sqrt(((nxx-nyy)/2)^2+nxy^2)
n2 = (nxx+nyy)/2 -sqrt(((nxx-nyy)/2)^2+nxy^2)
theta = 1/2*arctan(2*nxy/(nxx-nyy))
(nsx-nxx)*(nsy-nyy) >= nxy^2

Question 6

Use **separate sheets of paper** **for each question**!



Consider a rectangular membrane element (a by b) with four nodes in the corners. The displacement field is written as follows:

$$u_x(x,y) = n_1(x,y)\, u_{x1} + n_2(x,y)\, u_{x2} + n_3(x,y)\, u_{x3} + n_4(x,y)\, u_{x4}$$

$$u_y(x,y) = n_1(x,y)\, u_{y1} + n_2(x,y)\, u_{y2} + n_3(x,y)\, u_{y3} + n_4(x,y)\, u_{y4}$$

Additional functions are defined as:

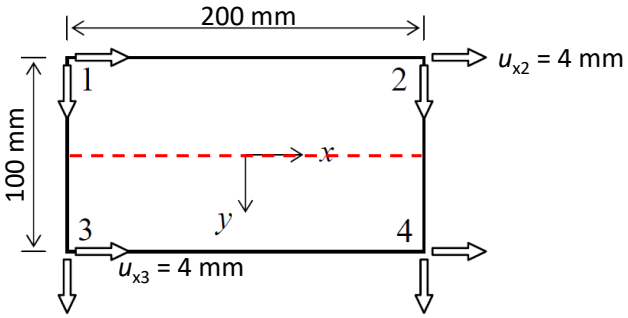
$$f^-(x) = ½ + x/a$$

$$g^-(y) = ½ + y/b$$

$$f^+(x) = ½ - x/a$$

$$g^+(y) = ½ - y/b$$

a) Provide a suitable set of shape functions by expressing them in $f^-(x)$, $g^-(y)$, $f^+(x)$ and $g^+(y)$.



Now consider the dimensions of the plate as indicated in the figure above. Also, consider that all nodal displacements are zero, except for $u_{x2} = u_{x3} = 4$ mm.

b) Sketch a graph of the strain ε_{xx} along the dashed line. Indicate values along the axes.

c) Similar for the strain ε_{yy}.

d) Similar for the strain γ_{xy}.

Brief answers

Question 1

a) $\sigma_{yy}^{D} < \sigma_{yy}^{E} < 0 < \sigma_{yy}^{R} \approx \sigma_{yy}^{Q}$ In D and E the stresses will be negative due to the normal force and the moment. In D the stresses will be most negative due to a singularity. The stresses in R and Q will be similar (and positive) due to the presence of the same axial force and moment.

b) $\sigma_{xx}^{G} < \sigma_{xx}^{E} </\approx \sigma_{xx}^{P} \approx \sigma_{xx}^{Q} \approx 0$ In P and Q the stresses will be close to zero (dynamic boundary condition). In G one principal stress will be negative and one principal stress will be close to zero. After rotation of 45° we will find that σ_{xx}^{G} (and σ_{yy}^{G}) is negative too. The element just above point E could make the stress in E negative too. But, since the rotation of the principal stresses will be marginal we may also expect a value close to zero in point E, like in P and G.

Question 2

a) $-t\int_{y_K}^{y_H} \sigma_{xy}\,\mathrm{d}y + F = 0$

b) $\int_{y_K}^{y_H} \sigma_{xx}\,\mathrm{d}y = 0$

c) $t\int_{y_K}^{y_H} (y + 3h + r)\sigma_{xx}\,\mathrm{d}y + hF = 0$ (around K) or

$t\int_{y_K}^{y_H} (y + 3h + r)\sigma_{xx}\,\mathrm{d}y + ht\int_{y_K}^{y_H} \sigma_{xy}\,\mathrm{d}y = 0$ (around J)

d) $-t\int_{y_L}^{y_M} \sigma_{xy}\,\mathrm{d}y + t\int_{y_M}^{y_N} \sigma_{yy}\,\mathrm{d}x - t\int_{y_N}^{y_O} \sigma_{xy}\,\mathrm{d}y + t\int_{y_O}^{y_G} \sigma_{yy}\,\mathrm{d}x + F = 0$

The first term will be (close) to zero.

Question 3

a) B
b) B
c) B
d) B
e) 5, 3
f) A, B, D
g) C

Question 4

a) Yes; the longest load path is from the curved edge to the fix support. This path is about 4 m. 4000/190 = 21 >> 5.
b) Increase, but only a little because the elements size is okay.
c) Yes. At the fixed and simply supported edges the deflection is zero. At the unsupported edges there is some deflection.
d) -115 kNm/m, however, this is in a singularity. Re-entrant corners. Y-direction.
e) Red and blue are in the singularities. Adjust the colour bar. (User defined smallest and largest value, for example -30 and 20 kNm/m).
f) Yes. In the fixed edge is a large myy. In the simply supported edges mxx = 0 (approximately). In the free edges the perpendicular moment are approximately zero. For example in the curved edge m2 is between -10 and +0.74 kNm/m. It should be zero. There is no reason to reject the results.
g) Cannot tell because the coordinate system of the mxy plot is not in the edge direction.
h) mxx + myy (or m1 + m2)

Question 5

```
> t := 210.·mm :
> nxx1 := -4300·kN/m : nyy1 := 620·kN/m : nxy1 := -1000·kN/m :
> nxx2 := 440·kN/m : nyy2 := -1180·kN/m : nxy2 := 860·kN/m :
> nsx := 827·kN/m : nsy := 827·kN/m :
> kN := 1000·N : m := 1000·mm : assume(N > 0) : assume(m > 0) :
> # a) ----------
> sxx1 := nxx1/t; syy1 := nyy1/t; sxy1 := nxy1/t;
```

$$sxx1 := -\frac{20.47619048\ N\sim}{mm\sim^2}$$

$$syy1 := \frac{2.952380952\ N\sim}{mm\sim^2}$$

$$sxy1 := -\frac{4.761904762\ N\sim}{mm\sim^2}$$

```
> sxx2 := nxx2/t; syy2 := nyy2/t; sxy2 := nxy2/t;
```

$$sxx2 := \frac{2.095238095\ N\sim}{mm\sim^2}$$

$$syy2 := -\frac{5.619047619\ N\sim}{mm\sim^2}$$

$$sxy2 := \frac{4.095238095\ N\sim}{mm\sim^2}$$

```
> # b) Yes because ----------
> (nsx - nxx1)·(nsy - nyy1) > nxy1^2;
```

$$\frac{1000000\ N\sim^2}{mm\sim^2} < \frac{1061289\ N\sim^2}{mm\sim^2}$$

```
> (nsx - nxx2)·(nsy - nyy2) > nxy2^2;
```

$$\frac{739600\ N\sim^2}{mm\sim^2} < \frac{776709\ N\sim^2}{mm\sim^2}$$

```
> # c) This is not possible because we would have to rotate over two angles simultainiously. ----
> theta1 = 1/2 * arctan(2. * nxy1/(nxx1 - nyy1)); # radians
```

$$\theta 1 = 0.1930503622$$

```
> theta2 = 1/2 * arctan(2. * nxy2/(nxx2 - nyy2));
```

$$\theta 2 = 0.4076646708$$

> # d) ----------

> $nsx1 := nxx1 + \text{abs}(nxy1);\ nsy1 := nyy1 + \text{abs}(nxy1);$

$$nsx1 := -\frac{3300\ N\sim}{mm\sim}$$

$$nsy1 := \frac{1620\ N\sim}{mm\sim}$$

> $nsx2 := nxx2 + \text{abs}(nxy2);\ nsy2 := nyy2 + \text{abs}(nxy2);$

$$nsx2 := \frac{1300\ N\sim}{mm\sim}$$

$$nsy2 := -\frac{320\ N\sim}{mm\sim}$$

> $nsxm := \max(0, nsx1, nsx2);$

$$nsxm := \frac{1300\ N\sim}{mm\sim}$$

> $nsym := \max(0, nsy1, nsy2);$

$$nsym := \frac{1620\ N\sim}{mm\sim}$$

> # e) Yes, for example this question. ----------

> $minimally_required := nsxm + nsym;$

$$minimally_required := \frac{2920\ N\sim}{mm\sim}$$

> $present_and_sufficient := nsx + nsy;$

$$present_and_sufficient := \frac{1654\ N\sim}{mm\sim}$$

Question 6

a) $n_1(x,y) = f^+(x)\, g^+(y)$, $n_2(x,y) = f^-(x)\, g^+(y)$, $n_3(x,y) = f^+(x)\, g^-(y)$, $n_4(x,y) = f^-(x)\, g^-(y)$

b) $\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0 + \frac{\partial n_2}{\partial x} u_{x2} + \frac{\partial n_3}{\partial x} u_{x3} + 0 = \frac{g^+}{a} u_{x2} + \frac{-g^-}{a} u_{x3} = \frac{1}{200}\left(\frac{1}{2} - \frac{y}{100}\right) 4 + \frac{-1}{200}\left(\frac{1}{2} + \frac{y}{100}\right) 4 = 0$

c) $\varepsilon_{yy} = 0$

d) $\gamma_{xy} = 0 - \frac{\partial n_2}{\partial y} u_{x2} + \frac{\partial n_3}{\partial y} u_{x3} + 0 + 0 + 0 + 0 + 0 = -\frac{f^-}{b} u_{x2} + \frac{f^+}{b} u_{x3} = -\frac{4}{100}\left(\frac{1}{2} - \frac{x}{200}\right) + \frac{4}{100}\left(\frac{1}{2} - \frac{x}{200}\right) = -\frac{4}{10000} x$

