Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Exam CIE4180: Plate Analysis
Thursday 31 January 2019, 13:30 – 16:30

This exam consists of 6 questions.
The respective weights are 2.2, 1.4, 1.5, 2.0, 1.5 and 1.4.

Use **separate sheets of paper for each question**!

Write your name and student number **on every page of your work.**

In this exam the notation and sign conventions of the reader (part 1 and 2) and the book (Plates and FEM) have been adopted. **You are supposed to use the same notation and sign convention.**

Question 1

Use **separate sheets of paper for each question**!

Consider the plane stress (or plate membrane) problem below. A plate has a length l, a height d and a thickness t.

The plate is supported at the two corners at the bottom and loaded by an equally distributed downwards line load p ("force per edge length") along the bottom edge. A horizontal constraint is added to avoid rigid body displacements.

The following material properties are used $E = 30000$ N/mm^2 , $\nu = 0.2$

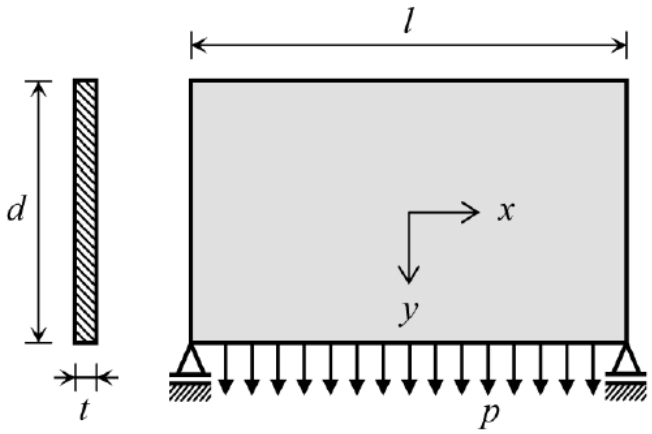


Figure 1.1: plane stress (or plate membrane) problem

Two geometrical cases are considered (the figure is not to scale!). The cases are labelled "Slender beam" and "Deep beam":

Table 1.1: Geometrical and loading properties

	l in mm	d in mm	t in mm	p in kN/m
Slender beam	30000	3000	1500	10
Deep beam	3000	3000	1500	10

An engineer analyses this problem with the following two finite element models.:

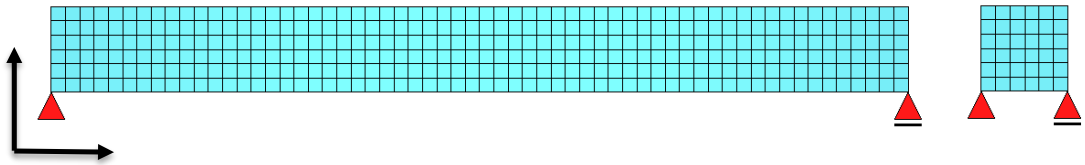


Figure 1.2: plane stress finite element models

The finite element program used a different axis system, denoted as x'y' **and possibly used a different sign conventions** than used in the course. The results of the program are denoted as $n_{x'x'}$ and $n_{y'y'}$.

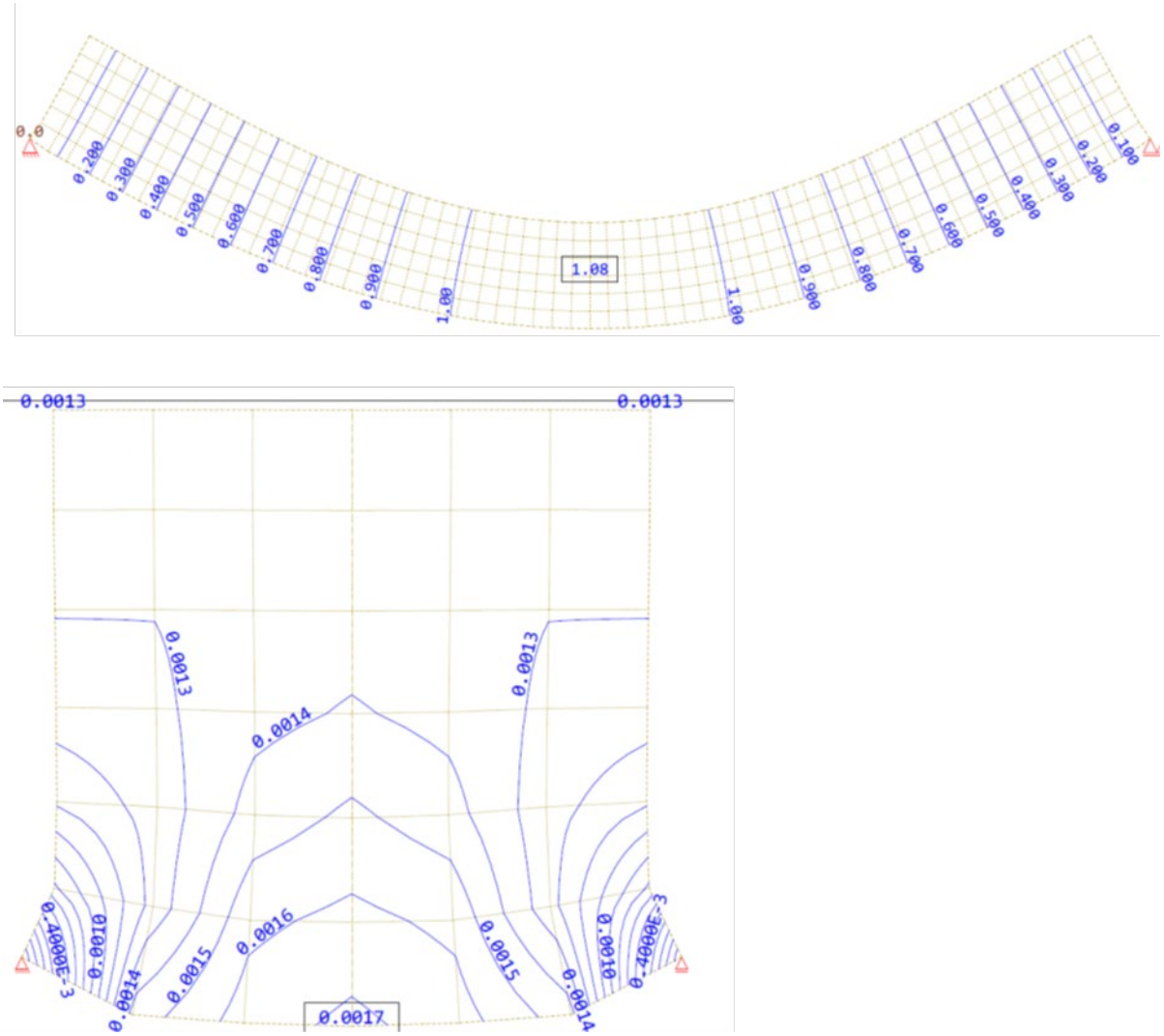


Figure 1.3: vertical displacements for both models in mm

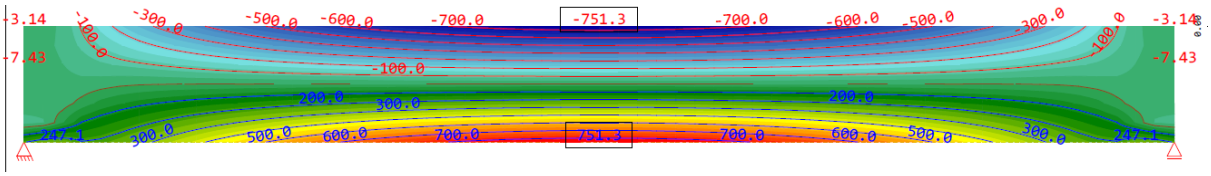


Figure 1.4: $n_{x'x'}$ contour plot for the slender beam in kN/m

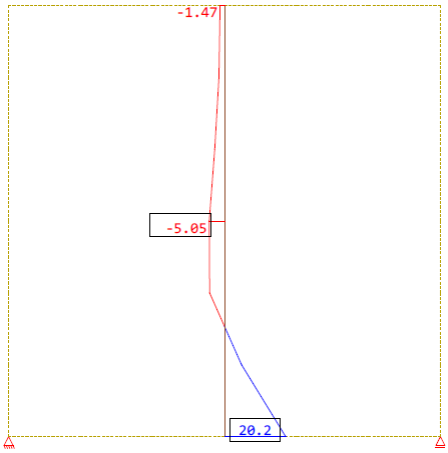


Figure 1.5: $n_{x'x'}$ diagram, along the vertical axis of symmetry for the deep beam in kN/m

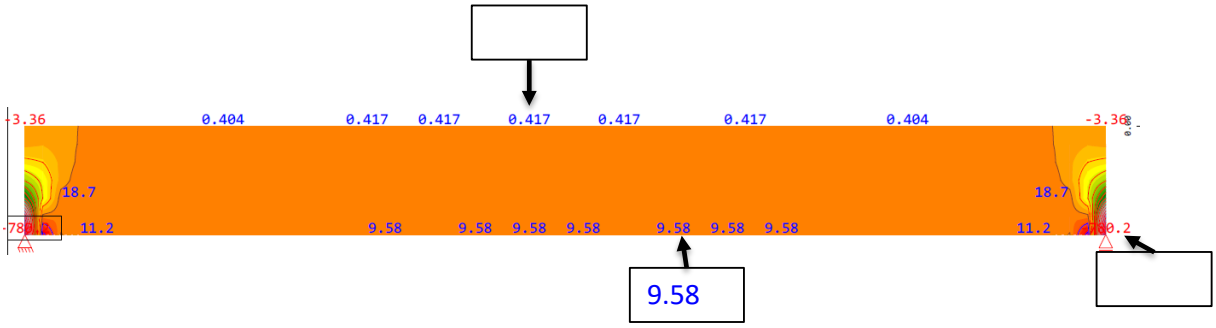


Figure 1.6: $n_{y'y'}$ contour plot for the slender beam in kN/m

The above Figures 1.3-6 provide finite element results for vertical displacements, $n_{x'x'}$ and $n_{y'y'}$.

- **a)** Did the engineer use rectangular elements with 4 nodes or with 8 nodes? Motivate your answer.
- **b)** Verify the value $n_{x'x'}$ = 751.3 kN/m in Figure 1.4.
- **c)** Does n_{xx} have the same sign as $n_{x'x'}$?
- **d)** How do you interpret the value $n_{x'x'}$ = -7.43 kN/m in Figure 1.4?
- **e)** How do you interpret the value $n_{x'x'}$ = -1.47 kN/m in Figure 1.5?
- **f)** What equilibrium checks are possible to verify the results of Figure 1.5? Shortly indicate how you would perform these checks.
- **g)** Does an (approximate) analytical solution exist to verify the result in Figure 1.5?
- **h)** How do you interpret the value $n_{y'y'}$ = 9.58 kN/m in Figure 1.6?
- **i)** Does n_{yy} have the same sign as $n_{y'y'}$?
- **j)** How do you interpret the value $n_{y'y'}$ = -780.2 kN/m in Figure 1.6?
- **k)** How do you interpret the value $n_{y'y'}$ = 0.417 kN/m in Figure 1.6?

Question 2

Use **separate sheets of paper for each question**!

A concrete plate with thickness t_c is covered with two steel sheets, both with thicknesses t_s, perfectly bonded on both faces of the concrete plate. The figure below shows a cross-section of this laminated concrete.

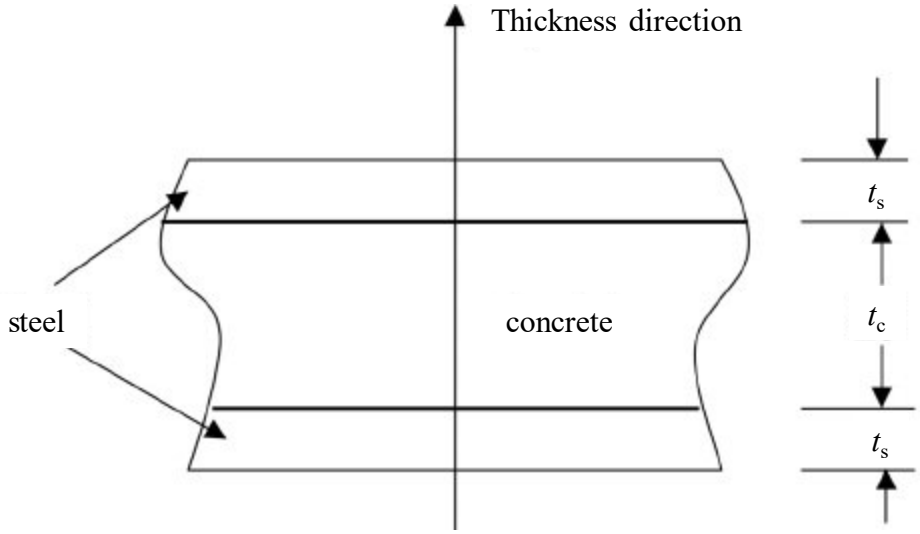


Figure 2.1: cross-section of the concrete plate covered by two fully-bonded steel sheets (figure not to scale!).

In an attempt to model this laminated concrete a small finite element model is made. In this finite element model, three 4-noded standard rectangular membrane elements are used that share the same four nodes:

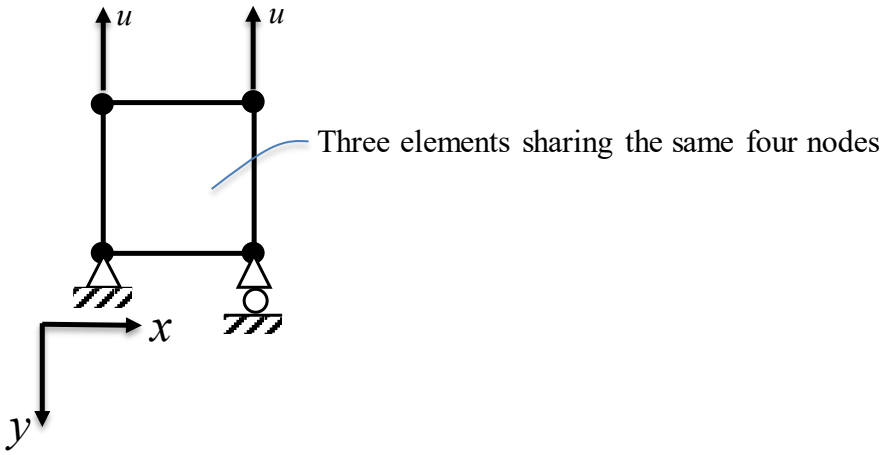


Figure 2.2: Finite element model with three elements, all sized 100 × 100 mm^2.

The concrete properties and thickness are: $E_c = 30000$ N/mm^2 , $\nu_c = 0.2$, $t_c = 50$ mm

The steel properties and thickness are: $E_s = 210000$ N/mm^2 , $\nu_s = 0.3$, $t_s = 1$ mm

The prescribed displacements are 1 mm. The size of the 3 elements is 100 × 100 mm^2.

a) What is the ratio of ${}^{c}\varepsilon_{yy} / {}^{s}\varepsilon_{yy}$, in which ${}^{c}\varepsilon_{yy}$ is the vertical strain in the middle of the concrete element and ${}^{s}\varepsilon_{yy}$ is the vertical strain in the middle of one of the steel elements.

b) Similarly, what is the horizontal stress ratio ${}^{c}\sigma_{xx} / {}^{s}\sigma_{xx}$?

Question 3

Use **separate sheets of paper** **for each question**!

a) Consider a triangular finite element with a node on each corner.
- Suppose that the element is for plates loaded in plane.
  How many shape functions does this element have?
- Suppose that the element is for plates loaded perpendicularly to their plane.
  How many shape functions does this element have?

b) What is the definition of the plate stiffness D ?

c) Many software developers think that we do not need to see membrane forces, moments or shear forces in plates. They think we only need to see the stresses.
Why this not true? Choose A, B, C or D.

A With stresses only we cannot check whether the boundary conditions are accurately fulfilled.
B Moments are required to design reinforcement in concrete plates.
C Membrane forces are required to check fatigue.
D Stresses can be different on the other side and it is inconvenient to turn a structure upside down many times.

d) Consider an interior molecule of the finite difference method of a plate that is loaded in plane.
What is the number of degrees of freedom in this molecule?

e) Which supposition is made in deriving the thin plate theory?
Choose from A, B, C and D. You can choose more than one.

A Straight needle remains straight. $u_x = z\varphi_x, u_y = z\varphi_y$
B There is no shear deformation. $\gamma_x = \gamma_y = 0$
C There is no stress perpendicular to the plate. $\sigma_{zz} = 0$
D The deflection is small. $w << t$

f) Who was Sophi Germain? Choose A, B, C or D.

A Somebody who gave money to science and in particular to the development of the thin plate theory.
B The mistress of Napoleon Bonaparte who handed out the award for developing the thin plate theory.
C The wife of Joseph Lagrange who derived the thin plate theory.
D The French scientist who derived the thin plate theory.

g) How many boundary conditions are needed for an edge of the …
… Navier equations?
… Kirchhoff equation?
… Mindlin-Reissner equations?

Question 4

Use **separate sheets of paper** **for each question**!

A plate is loaded perpendicularly to its plane. Three edges are simply supported and the others are free (Fig. 6). The plate length is 16 m and the width is 14 m. The plate thickness is 200 mm. A distributed load of 7 kN/m^2 is applied in the z-direction. This distributed load includes self-weight.
A linear elastic finite element model is made. The following material properties are used, E = 34100 N/mm^2, ν = 0.2. Four-node plate elements are used (Kirchhoff theory). The element size is 400 mm. The results are shown in Figures 4.1 to 4.6.

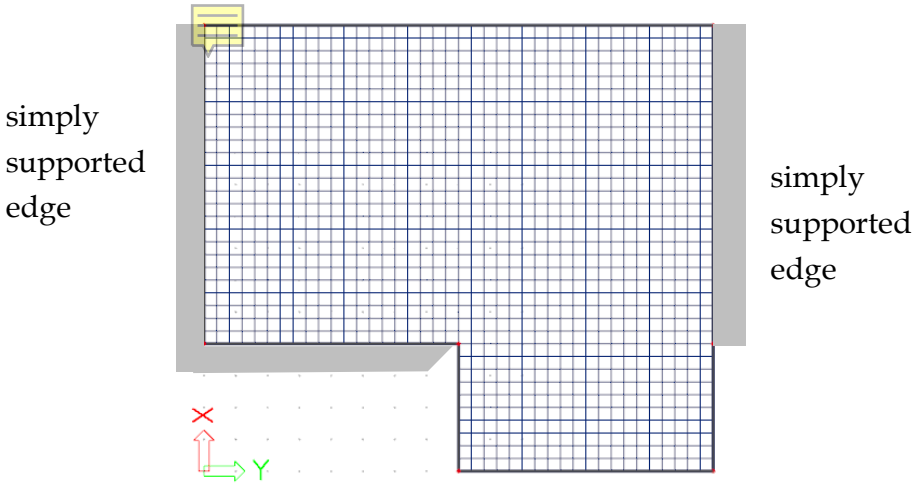


Figure 4.1. Plate with supports, load and finite element mesh

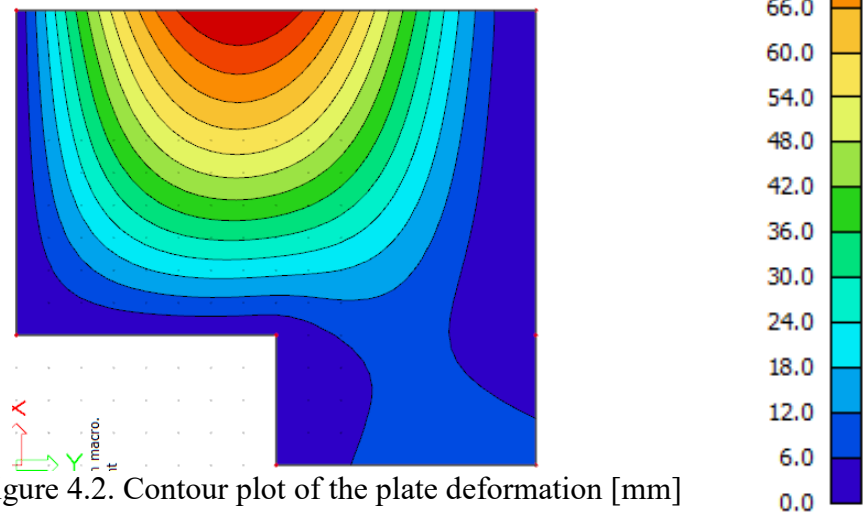


Figure 4.2. Contour plot of the plate deformation [mm]

a) Is the element size acceptable? Explain your answer.

b) How can we improve the accuracy?

c) Consider the deflection in Figure 4.2. Is it towards you or away from you? Explain your answer.

d) Does the deformation agree with the boundary conditions? Explain your answer.

e) Where in this plate can we expect singularities?

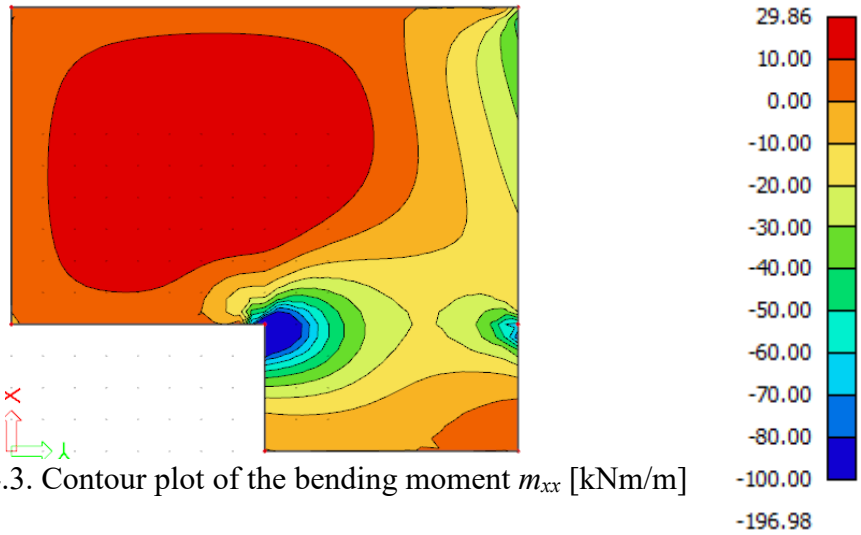


Figure 4.3. Contour plot of the bending moment m_{xx} [kNm/m]

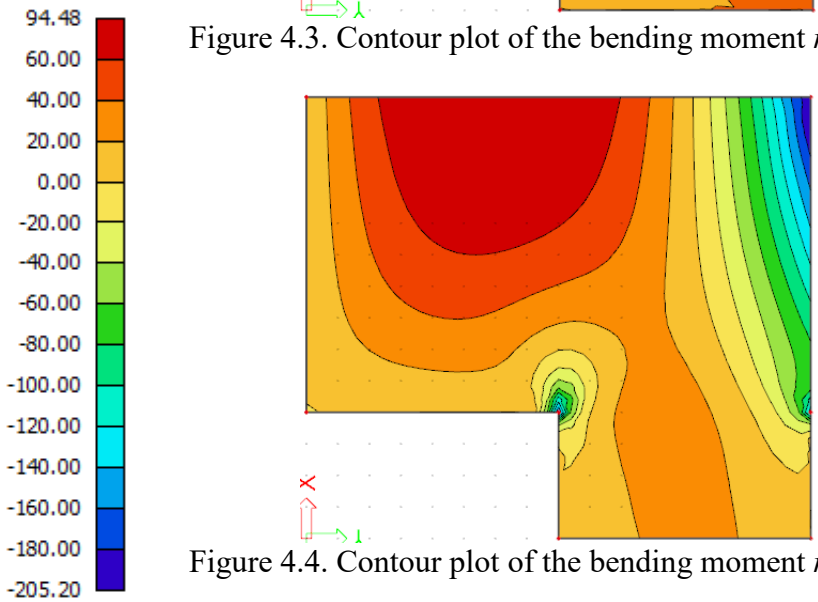


Figure 4.4. Contour plot of the bending moment m_{yy} [kNm/m]

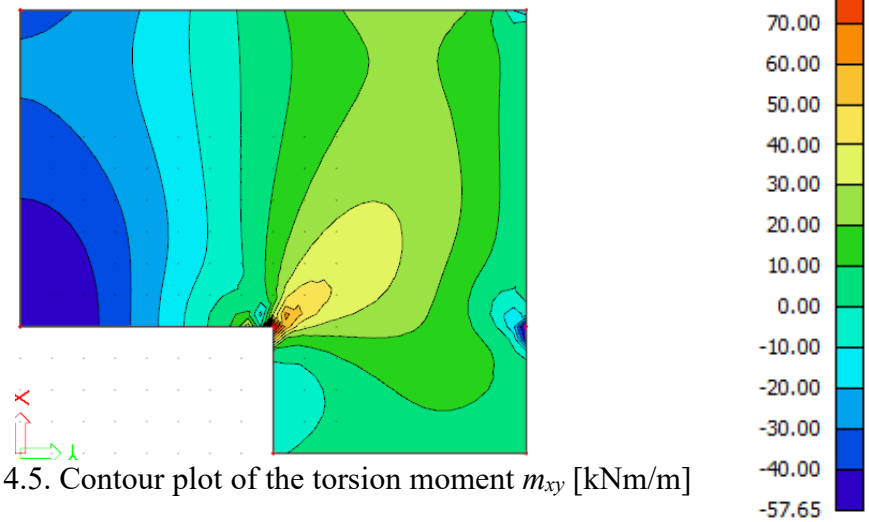


Figure 4.5. Contour plot of the torsion moment m_{xy} [kNm/m]

f) Do the moments agree with the boundary conditions? Explain your answer.

g) What is the largest value of the concentrated shear force V in the plate edge?

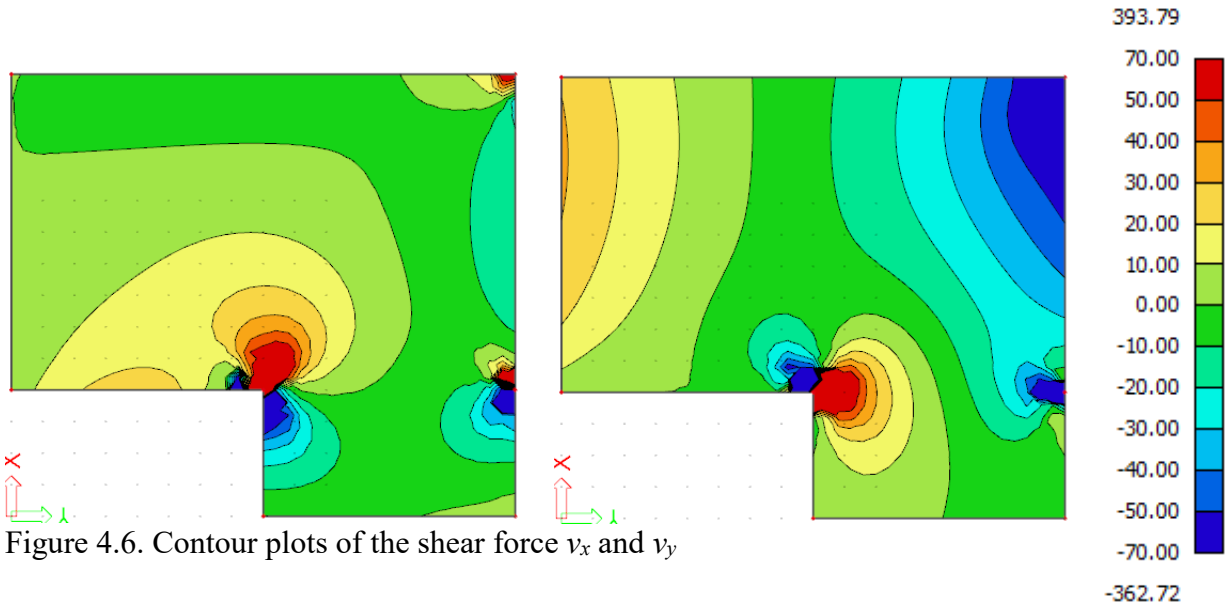


Figure 4.6. Contour plots of the shear force v_x and v_y

h) What happens in the right top corner of the plate?
(Suggestion; compare Figure 4.3 and Figure 4.6.)

Question 5

Use **separate sheets of paper** **for each question**!

A reinforced concrete plate has a thickness of 310 mm. The plate carries 11 kN/m^2 (self-weight plus live load; partial factors included). At a particular location the moments are
m_{xx} =111 kNm/m,
m_{yy} = -42 kNm/m,
m_{xy} = -59 kNm/m.

The reinforcement consists of two meshes. As usual, one mesh is close to the bottom surface the other mesh is close to the top surface. Each mesh has bars in the *x* direction and bars in the *y* direction. All bars have a diameter of 16 mm and are spaced 120 mm. The yield strength of the reinforcing steel is 380 N/mm^2 (partial factor included). The moment capacity due to the bars in the *x* direction is 141 kNm/m. The moment capacity due to the bars in the *y* direction is 131 kNm/m.

a) Are the reinforcing bars in the principal moment directions? How do we know?

b) Suppose that the plate is linear elastic. What are the normal stresses and shear stresses?

c) Calculate the reinforcement moment capacities that are minimally required in the *x* and *y* directions.

d) Is the reinforcement sufficient? Explain your answer.

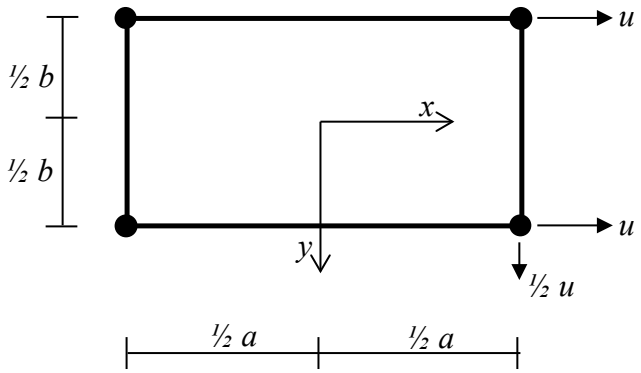
e) Suppose that there are also membrane forces and out of plane shear forces in the plate. Which model can be used to check or design the reinforcement?

Useful equations
top $m_{sxt} = -m_{xx}+|m_{xy}|$ $m_{syt} = -m_{yy}+|m_{xy}|$
bottom $m_{sxb} = m_{xx}+|m_{xy}|$ $m_{syb} = m_{yy}+|m_{xy}|$
$m_1=(m_{xx}+m_{yy})/2+\text{sqrt}(((m_{xx}-m_{yy})/2)^2+m_{xy}^2)$
$m_2=(m_{xx}+m_{yy})/2 -\text{sqrt}(((m_{xx}-m_{yy})/2)^2+m_{xy}^2)$
top $(m_{sxt}+m_{xx})*(m_{syt}+m_{yy}) >= m_{xy}^2$
bottom $(m_{sxb}-m_{xx})*(m_{syb}-m_{yy}) >= m_{xy}^2$

Question 6

Use **separate sheets of paper** **for each question**!

Consider a single standard rectangular membrane element with the following three nodal displacements (u, u and $½u$):



All other nodal displacements are zero!

a) Give an expression for the displacement field $u_x(x, y)$ and for strain field $\varepsilon_{xx}(x, y)$.

b) Give an expression for the displacement field $u_y(x, y)$ and for strain field $\varepsilon_{yy}(x, y)$.

c) Is the stress field $\sigma_{xx}(x, y)$ depending on x?

Brief answers

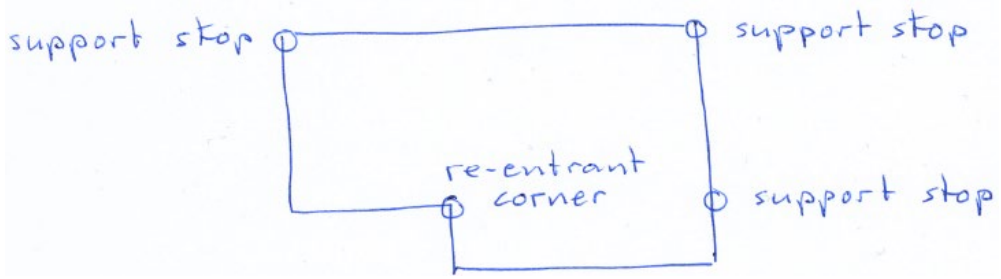
Question 1
a) 4 nodes. See linear displacement interpolation in Figure 1.3: the edges of the elements stay straight.
b) Standard beam theory applies for this beam. $M = \frac{pl^2}{8}; W = \frac{td^2}{6};\ n_{xx} = t\frac{M}{W} = 750$ kN/m
c) Yes
d) At the right-hand edge $n_{x'x'}$ should be zero. The difference is caused by the finite element approximation of the dynamic boundary conditions.
e) Just as the outcome of the analysis. It is small but not zero
f) Horizontal equilibrium by integrating the $n_{x'x'}$ over y and verify if it is close to zero. Moment equilibrium by integrating $an_{x'x'}$ over y, in with a is the distance to the neutral line, and verify if this is close to M.
g) Yes, see reader
h) As $n_{y'y'} = 9.58$ which is close to p. The difference is caused by the finite element approximation of the dynamic boundary conditions.
i) Yes
j) As a value which will strongly depend on the element size and of no practical relevance.
k) At the top edge $n_{y'y'}$ should be zero. The value is acceptably close to zero.

Question 2
a) 1 (same displacement fields, same strain fields)
b) -0.04 (from equilibrium $t_c\,{}^c\sigma_{xx} + 2\,t_s\,{}^s\sigma_{xx} = 0$)

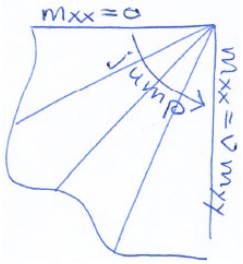
Question 3
a) 6, 9
b) $\frac{Et^3}{12(1-\nu^2)}$
c) B
d) 9
e) A, B, C and D
f) D
g) 2, 2, 3

Question 4
a) Yes. 400 mm is two times the plate thickness of 200 mm. This is the recommended element size for the Kirchhoff theory.
b) By choosing smaller elements.
c) Away from me. The displacements are positive (see legend). Positive displacements are in the z direction (definition). The z axis points away from me (right-hand coordinate system). Conclusion: The deflection is away from me.
Note that the displacement is consistent with the loading, which is also in the z direction.
d) Yes. On all simply supported edges the displacement is zero. (half a point)
No. On all simply supported edges the displacement is zero, however, on the right- hand edge the slope of the plate is much smaller than on the left hand side. There is no cause for the plate to behave so differently on these edges. Something is wrong. (full point)

e)



f) No, on the right-hand edge moment myy should be zero.
g) infinite (it is in a singularity).
h) Singularity. Figure 8, right top corner, shows a jump in mxx. It jumps from 0 on the free edge to some value along the (apparently) fixed edge. Since, $v_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y}$, a jump in mxx produces an infinite shear force vx.



Question 5
a) No, mxy is not equal to 0
b) Top sxxt = -6*mxx/t^2 = -6* 111000/310^2 = -6.93 N/mm^2
syyt = -6*myy/t^2 = -6*(-42000)/310^2 = 2.62 N/mm^2
sxyt = -6*mxy/t^2 = -6*(-59000)/310^2 = 3.68 N/mm^2
Bottom sxxb = 6*mxx/t^2 = 6* 111000/310^2 = 6.93 N/mm^2
syyb = 6*myy/t^2 = 6*(-32000)/310^2 = -2.62 N/mm^2
sxyb = 6*mxy/t^2 = 6*(-59000)/310^2 = -3.68 N/mm^2
c) Top msxt = -mxx+|mxy| =-111+59 = -52 kNm/m = 0
msyt = -myy+|mxy| = 42+59 = 101 kNm/m
Bottom msxb = mxx+|mxy| = 111+59 = 170 kNm/m
msyb = myy+|mxy| = -42+59 = 17 kNm/m
d) Top (msxt+mxx)*(msyt+myy) >= mxy^2
(141+111)*(131-42)>=?(-59)^2
22500>=?3481
Bottom (msxb-mxx)*(msyb-myy) >= mxy^2
(141-111)*(131+42)>=?(-59)^2
5254>=?3481
More than sufficient capacity
e) Three layer sandwich model

Question 6
a) $u_x(x,y) = \frac{1}{2}u + \frac{u}{a}x;\ \varepsilon_{xx}(x,y) = \frac{u}{a}$
b) $u_y(x,y) = \frac{1}{8}u + \frac{1}{4}\frac{u}{a}x + \frac{1}{4}\frac{u}{b}y + \frac{1}{2}\frac{u}{ab}xy;\ \varepsilon_{yy}(x,y) = \frac{1}{4}\frac{u}{b} + \frac{1}{2}\frac{u}{ab}x;$
c) Yes, since ε_{yy} is a function of x