Plate analysis, theory and applications

Lesson: Plates loaded in their plane

CIE4180

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The majority of the slides have been prepared by Xueyan Liu, Cor Kasbergen & Alexander Schmetz

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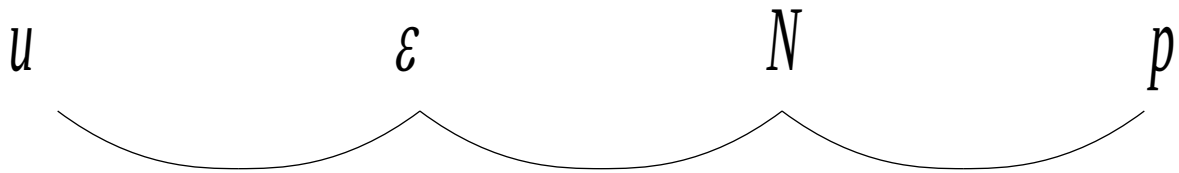
Delft University of Technology

Uniaxially loaded plate



kinematic equation

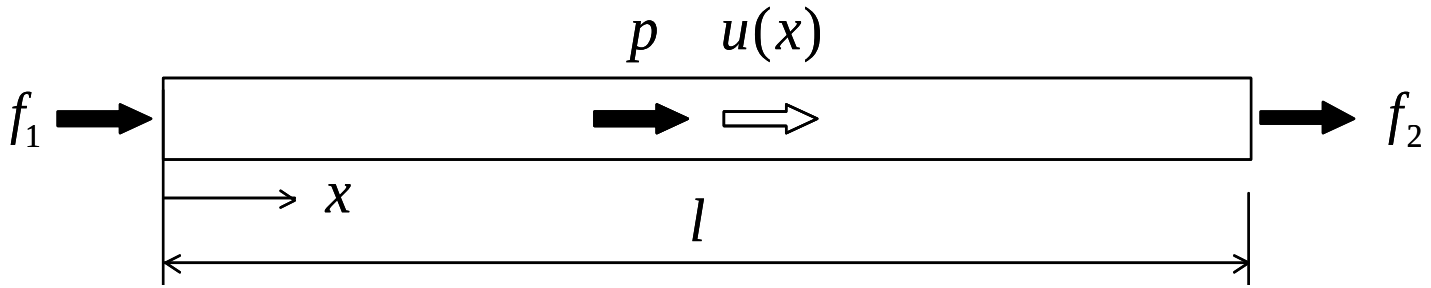
constitutive equation

equilibrium equation

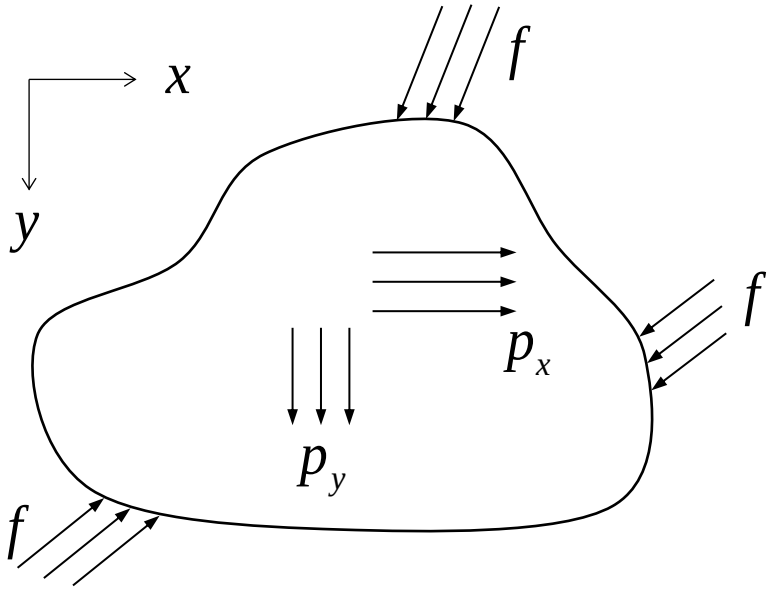
$$\varepsilon = \frac{du}{dx}$$

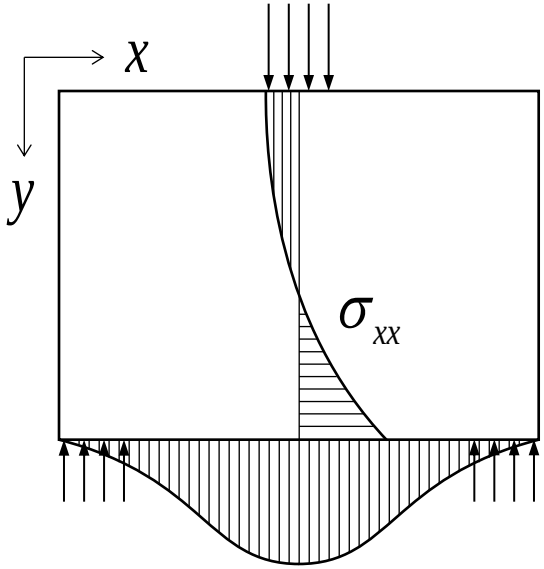
$$N = EA\varepsilon$$

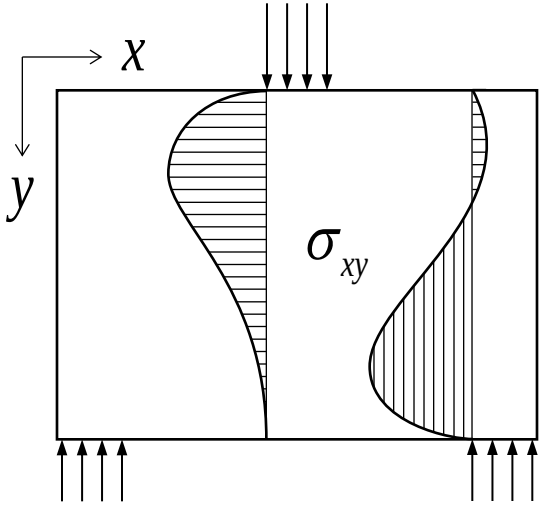
$$-\frac{dN}{dx} = p$$



Thin flat plates loaded in-plane

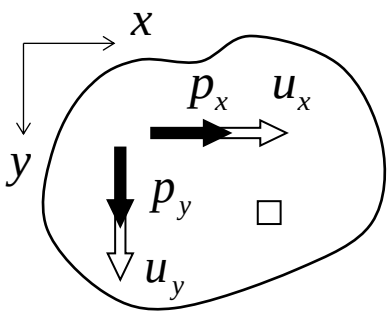


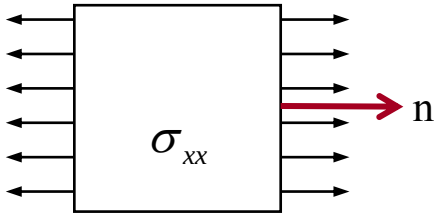


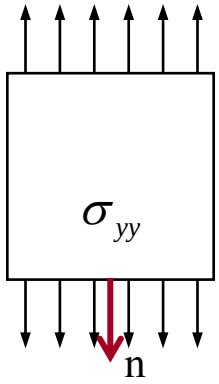


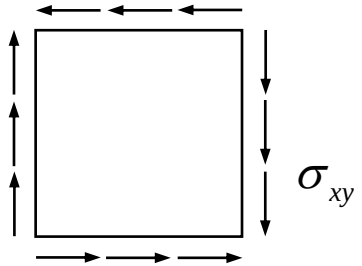
A plate loaded in-plane

Signs :





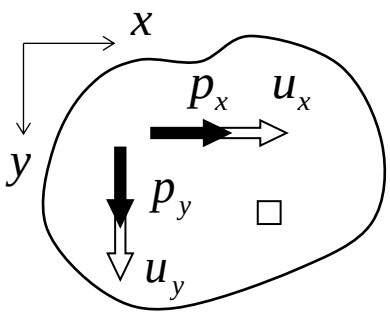


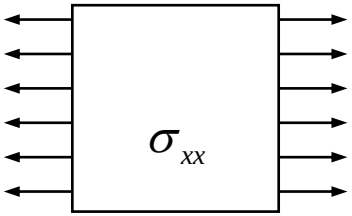


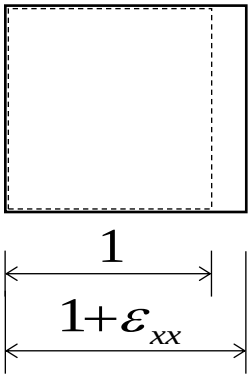
A stress is positive if it acts in the positive coordinate direction on a plane with the normal vector in the positive coordinate direction. Correspondingly, a stress is positive if it acts in the negative coordinate direction on a plane with its normal in the negative coordinate direction.

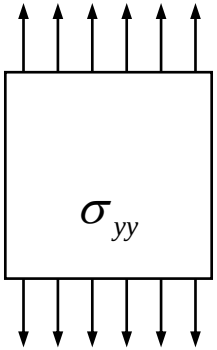
A plate loaded in-plane

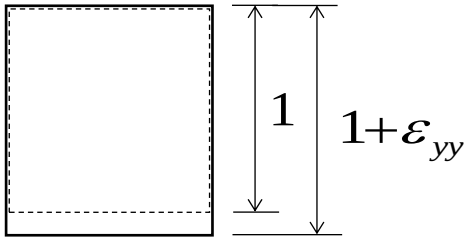
Signs :

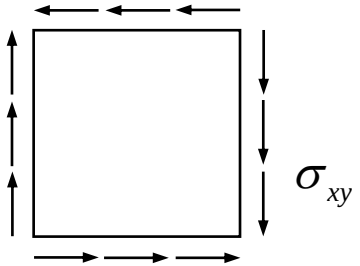


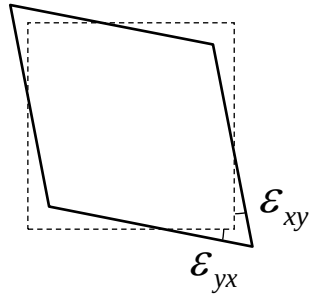






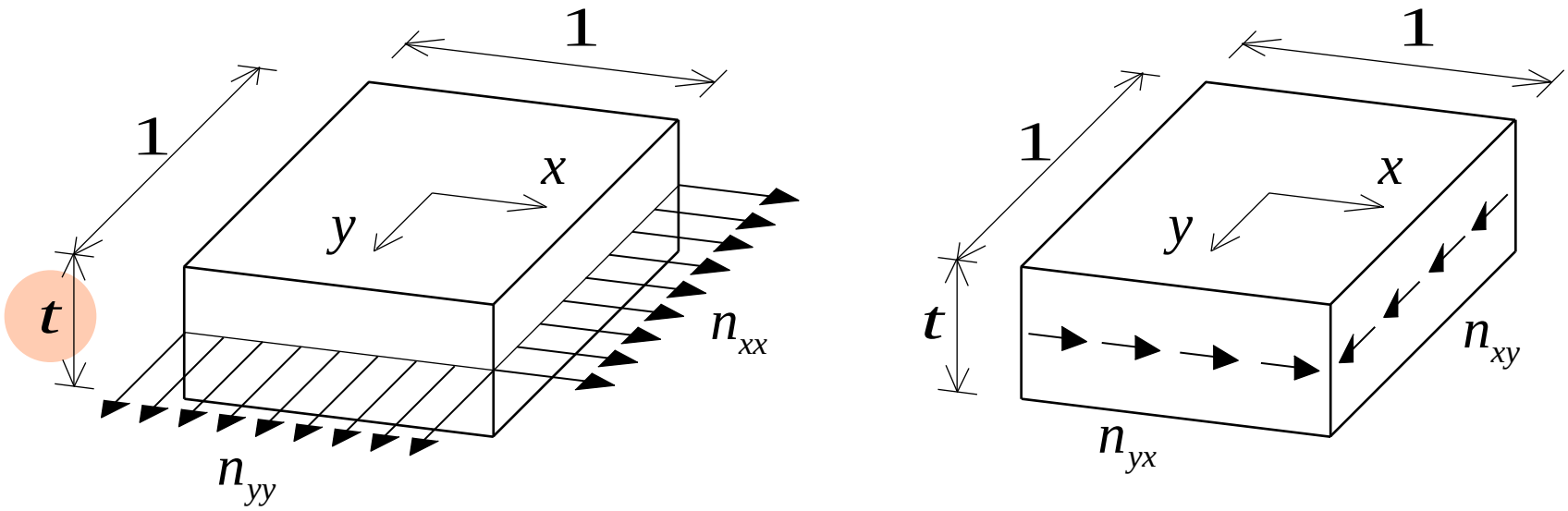






$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx}$$

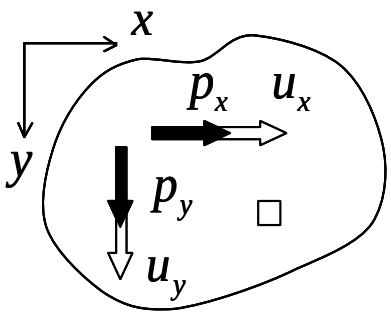
A plate loaded in-plane

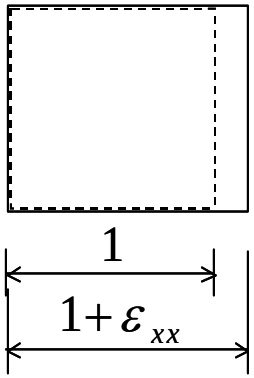


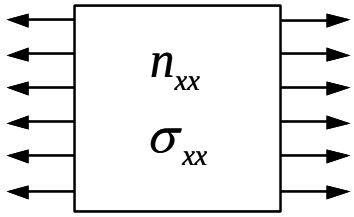
$$n = \sigma t$$

Membrane force

A plate loaded in-plane





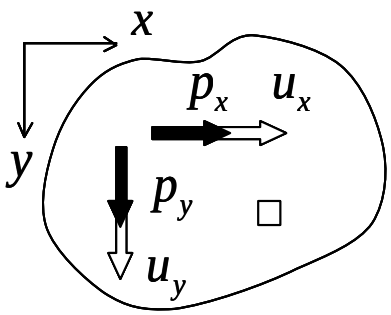


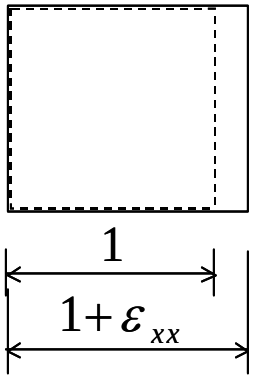
Derive:

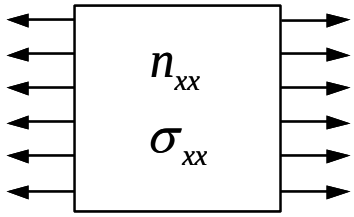
$$\boldsymbol{u}=\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \boldsymbol{n}=\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p}=\begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

kinematic equations *constitutive equations* *equilibrium equations*

A plate loaded in-plane







Derive:

$$\boldsymbol{u}=\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \boldsymbol{n}=\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p}=\begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

kinematic equations | *constitutive equations* | *equilibrium equations*

A plate loaded in-plane

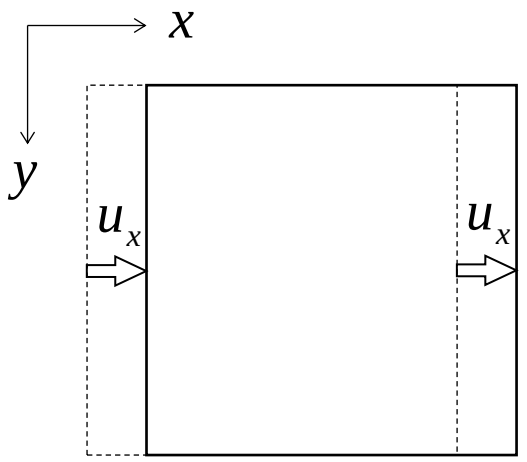
Kinematic equations:

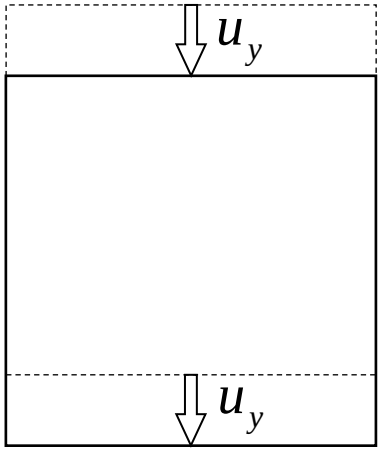
3 rigid body displacements +

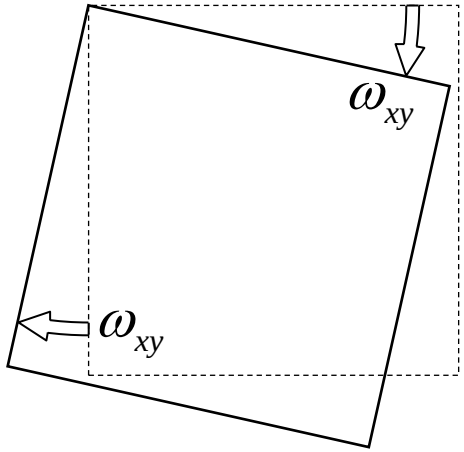
3 deformations

$$\boldsymbol{u}=\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \omega_{xy}$$

- rigid body displacemets





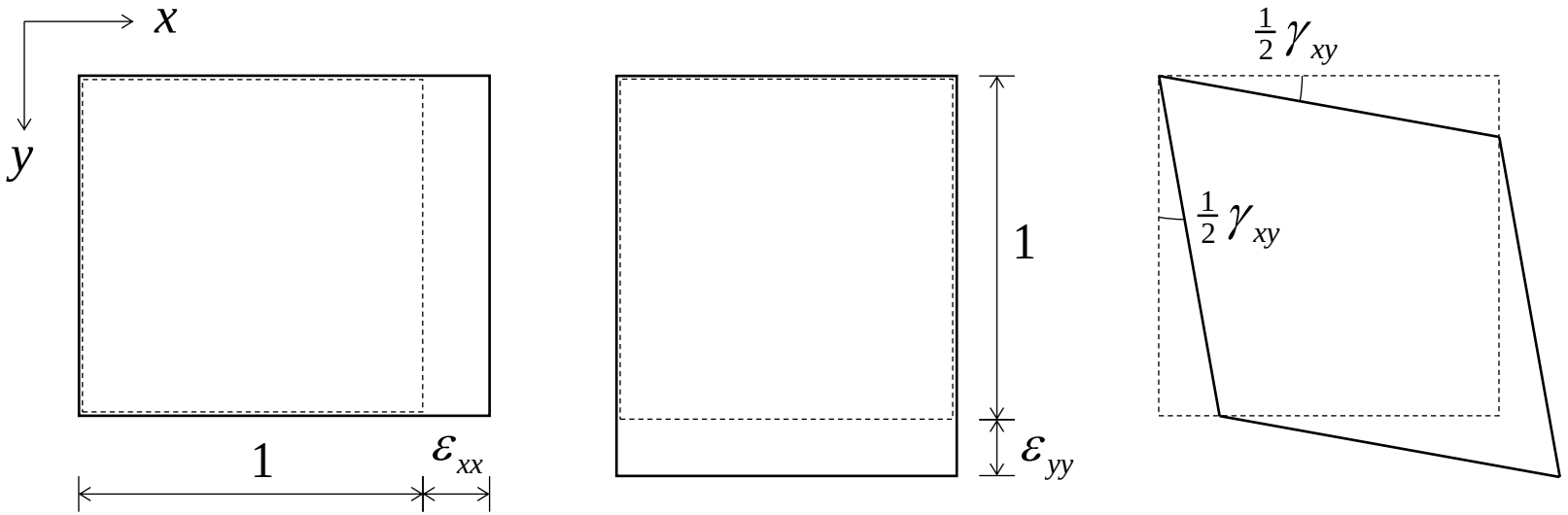


A plate loaded in-plane

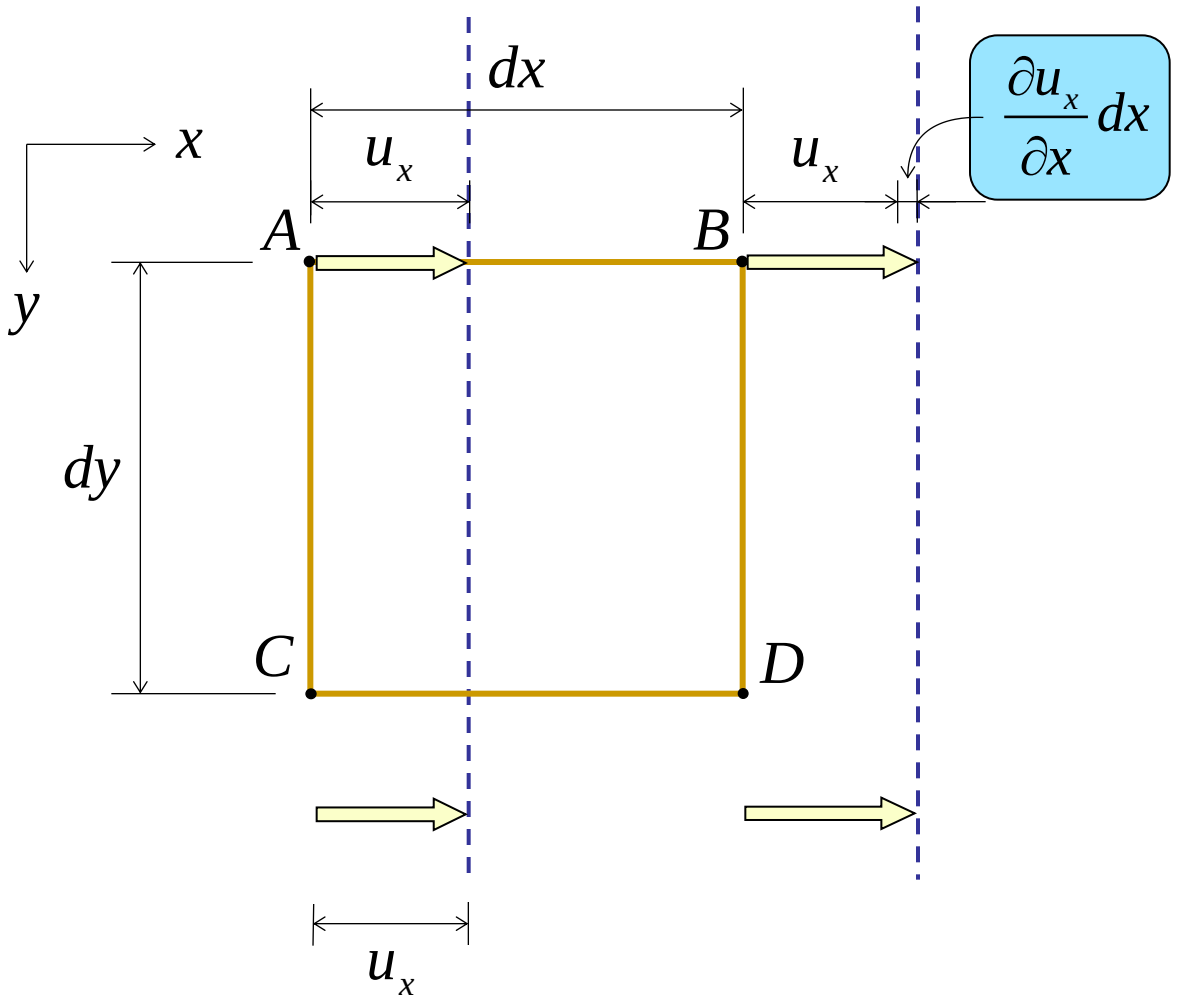
Kinematic equations:

- deformations

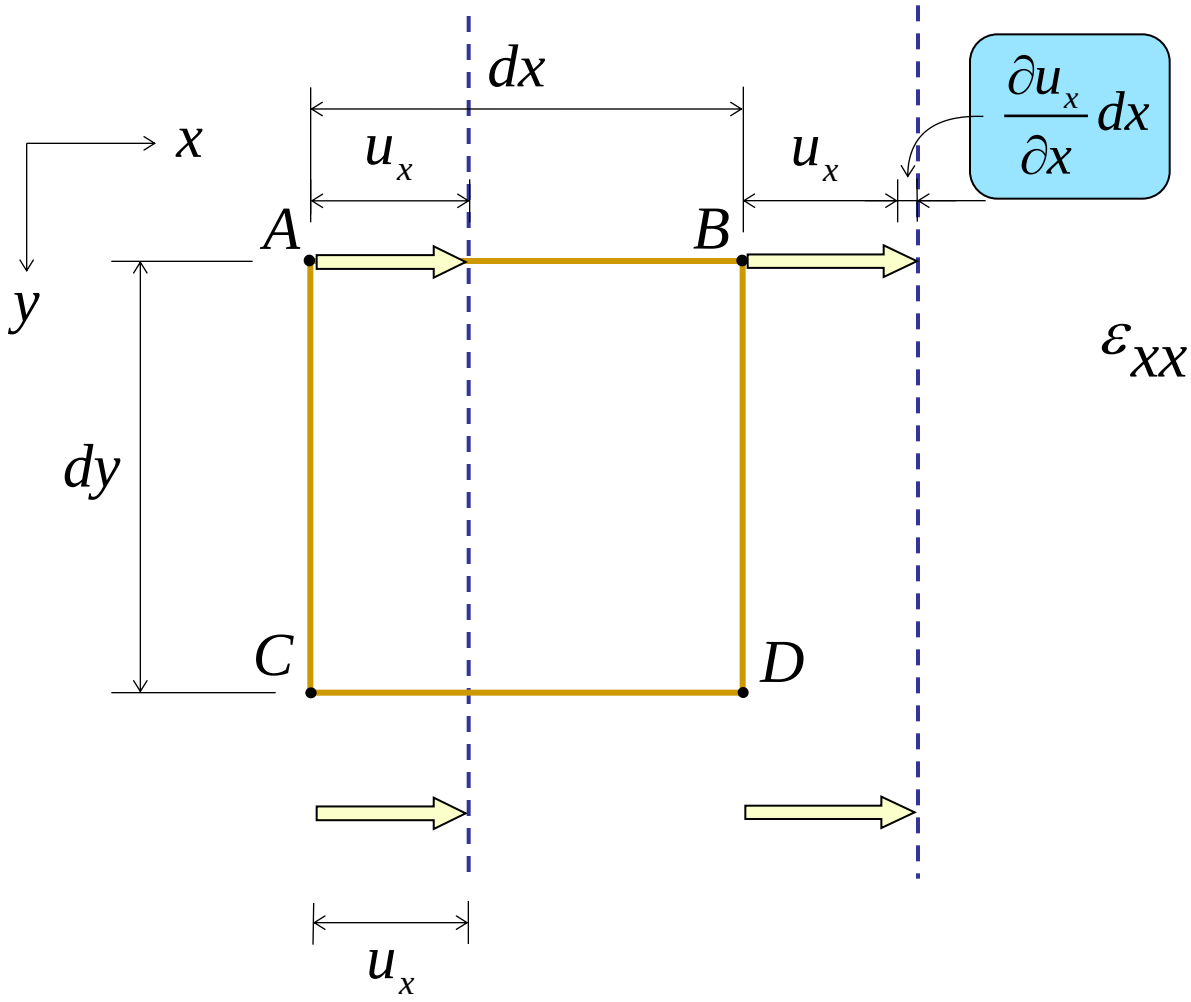
$$\boldsymbol{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$



displacements → kinematic equations → strains

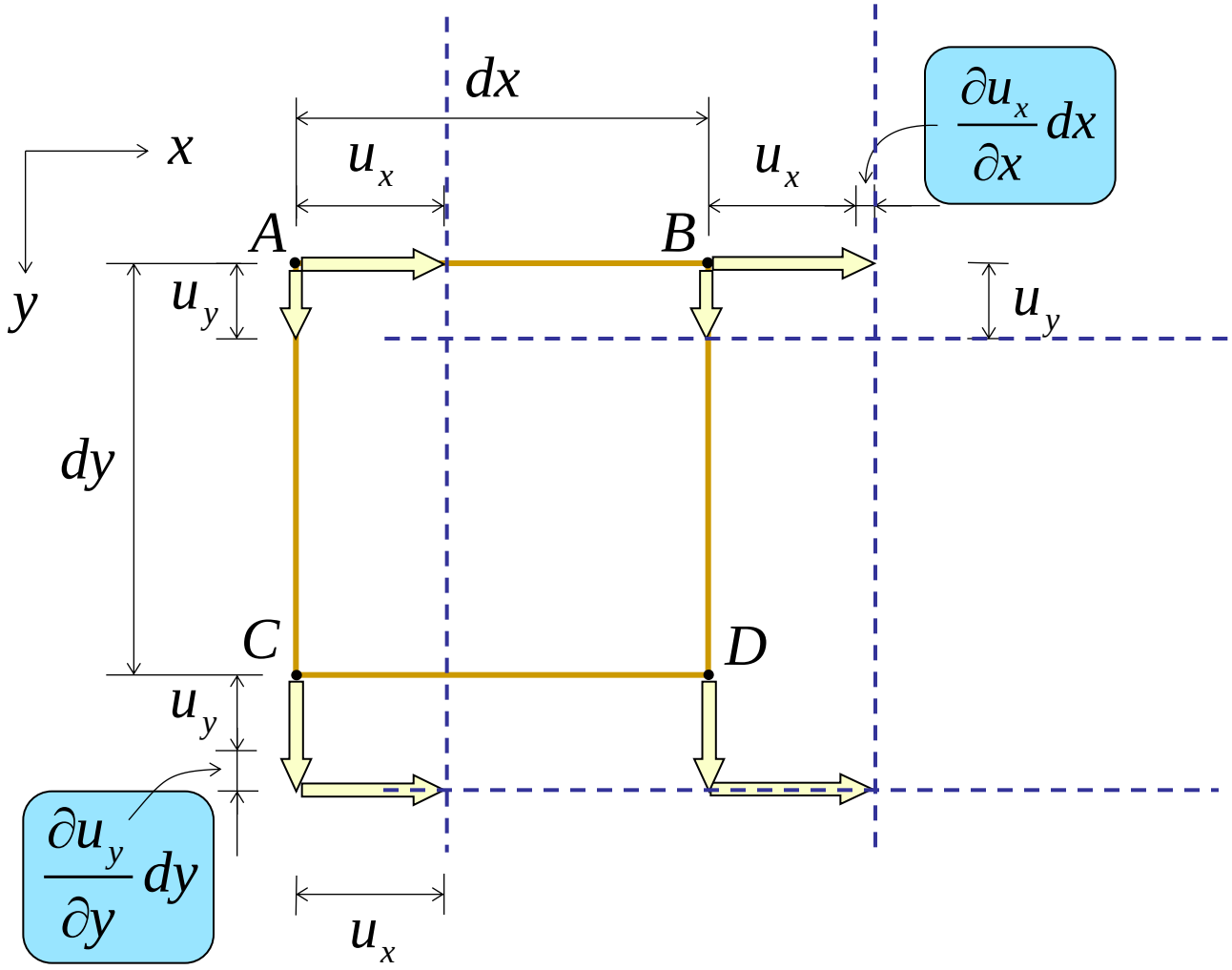


displacements → kinematic equations → strains

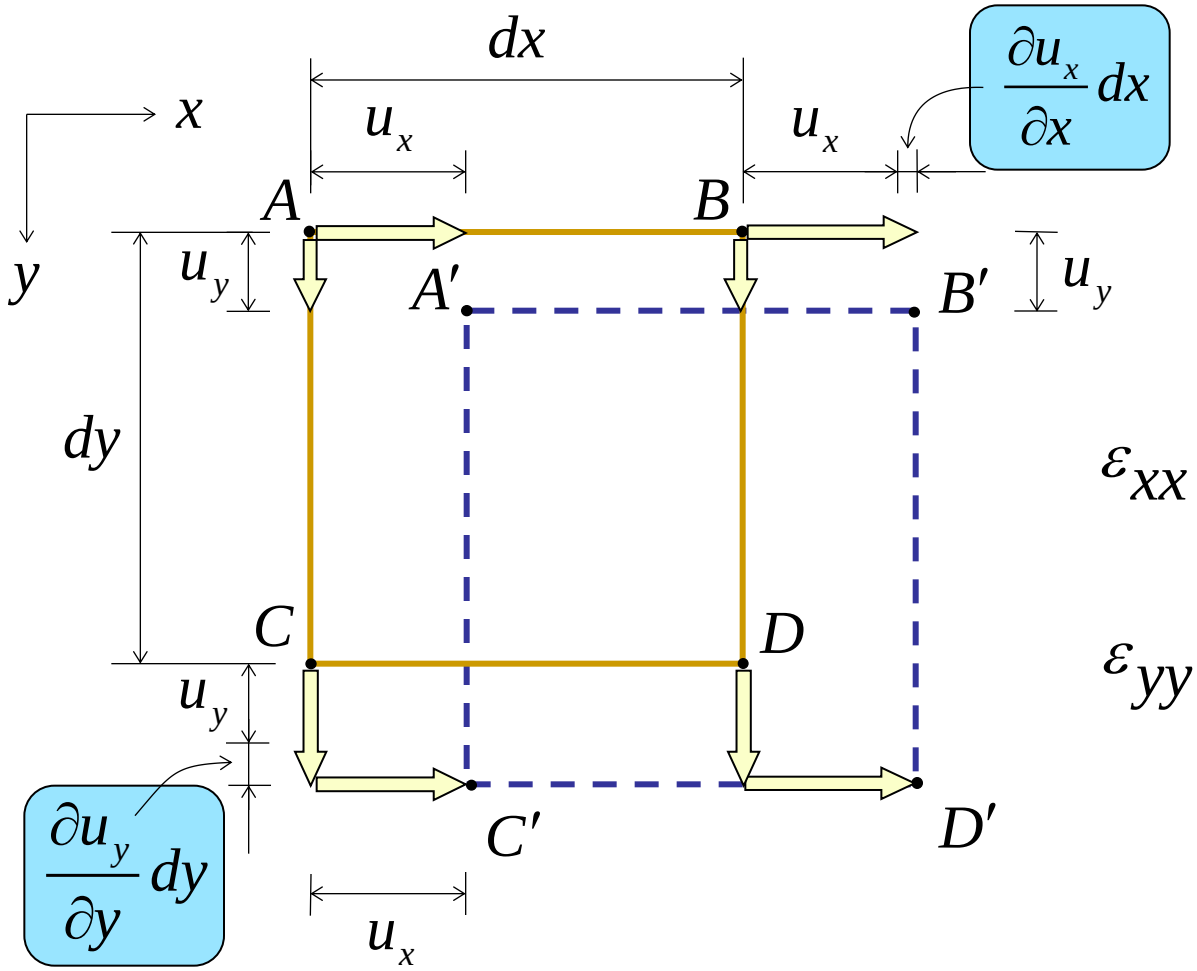


$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

displacements → kinematic equations → strains



displacements → kinematic equations → strains

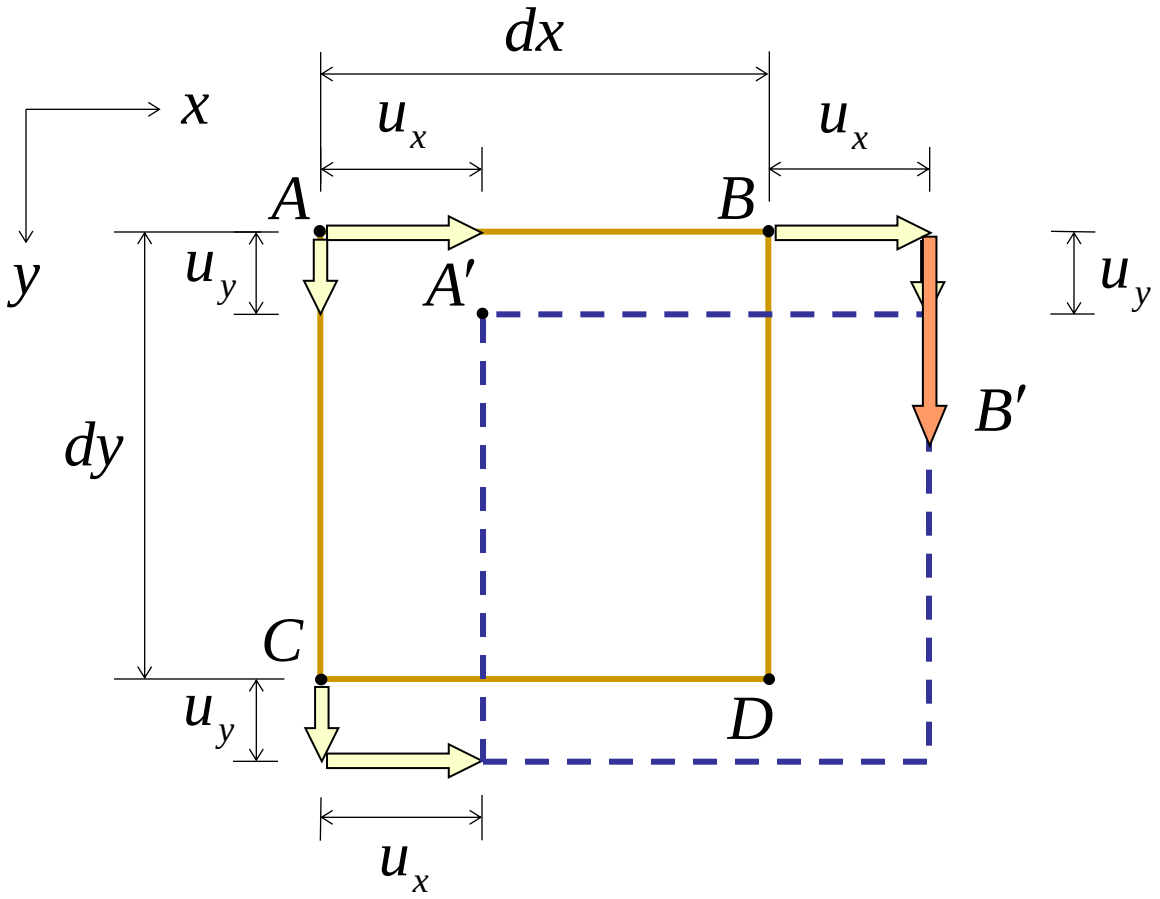


$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

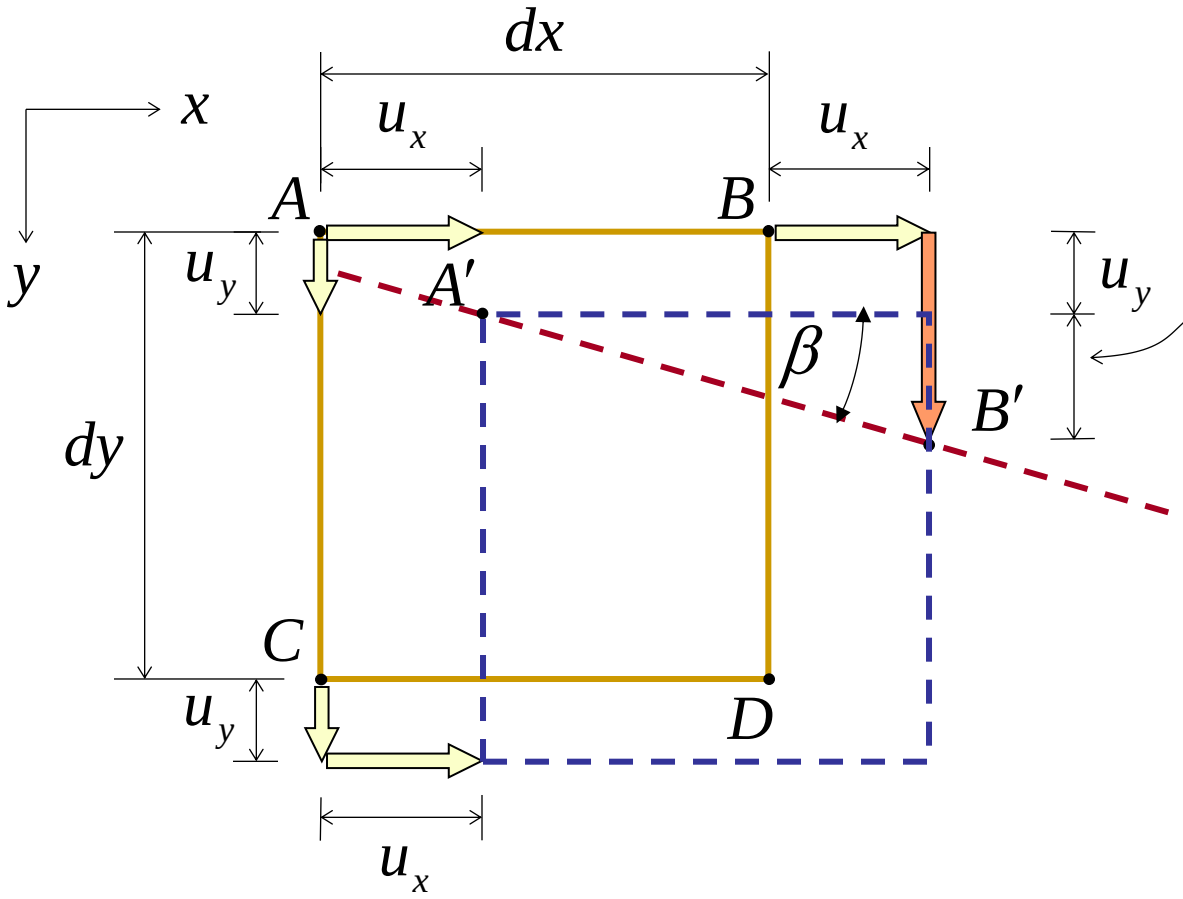
A plate loaded in-plane

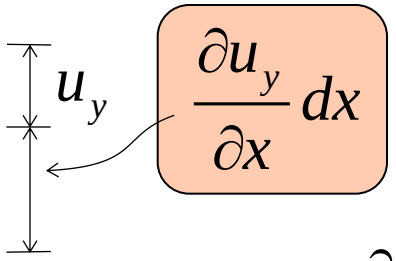
- shear strain



A plate loaded in-plane

- shear strain

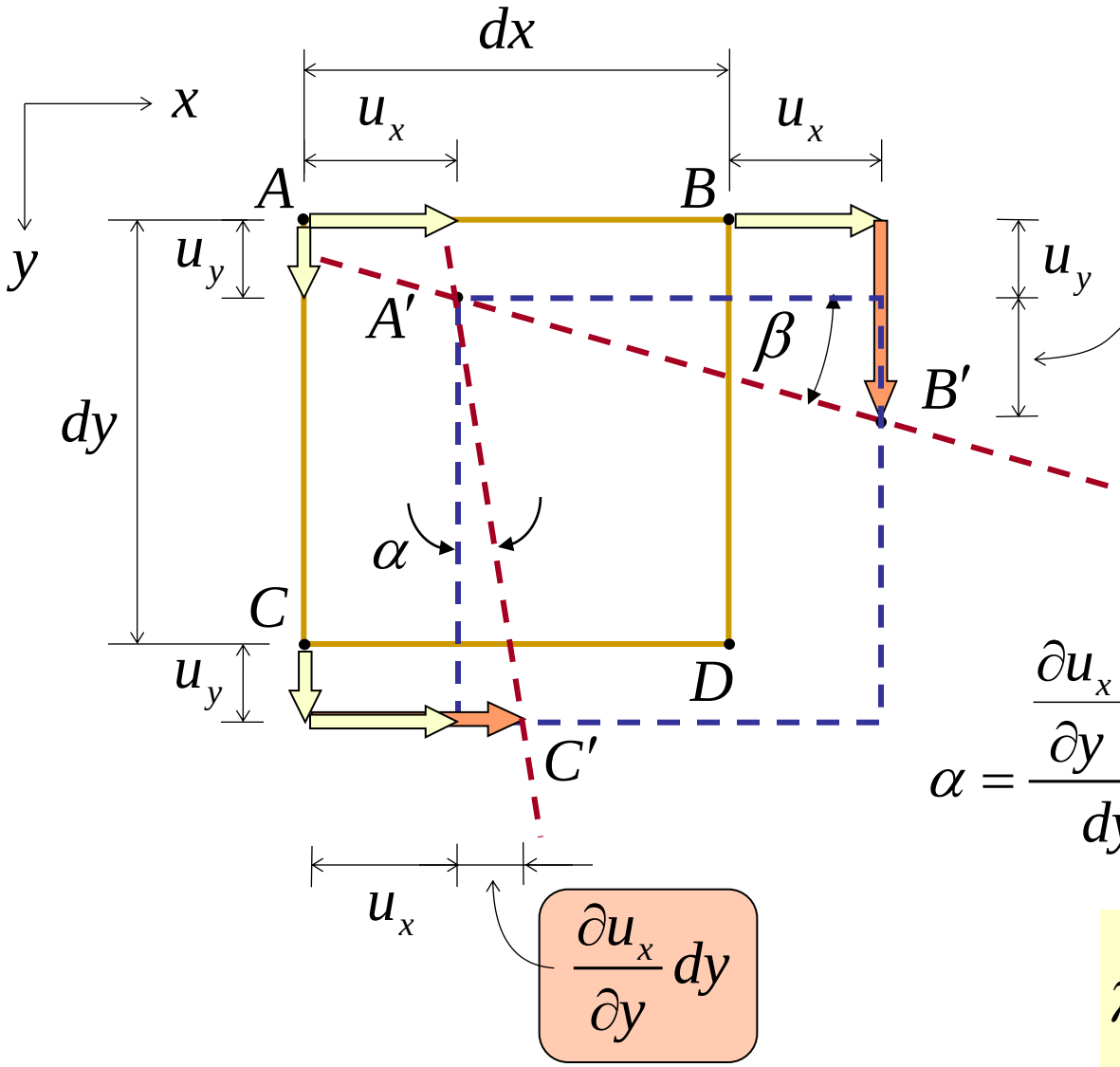




$$\beta = \frac{\frac{\partial u_y}{\partial x} dx}{dx} = \frac{\partial u_y}{\partial x}$$

A plate loaded in-plane

- shear strain



$\frac{\partial u_y}{\partial x} dx$

u_y

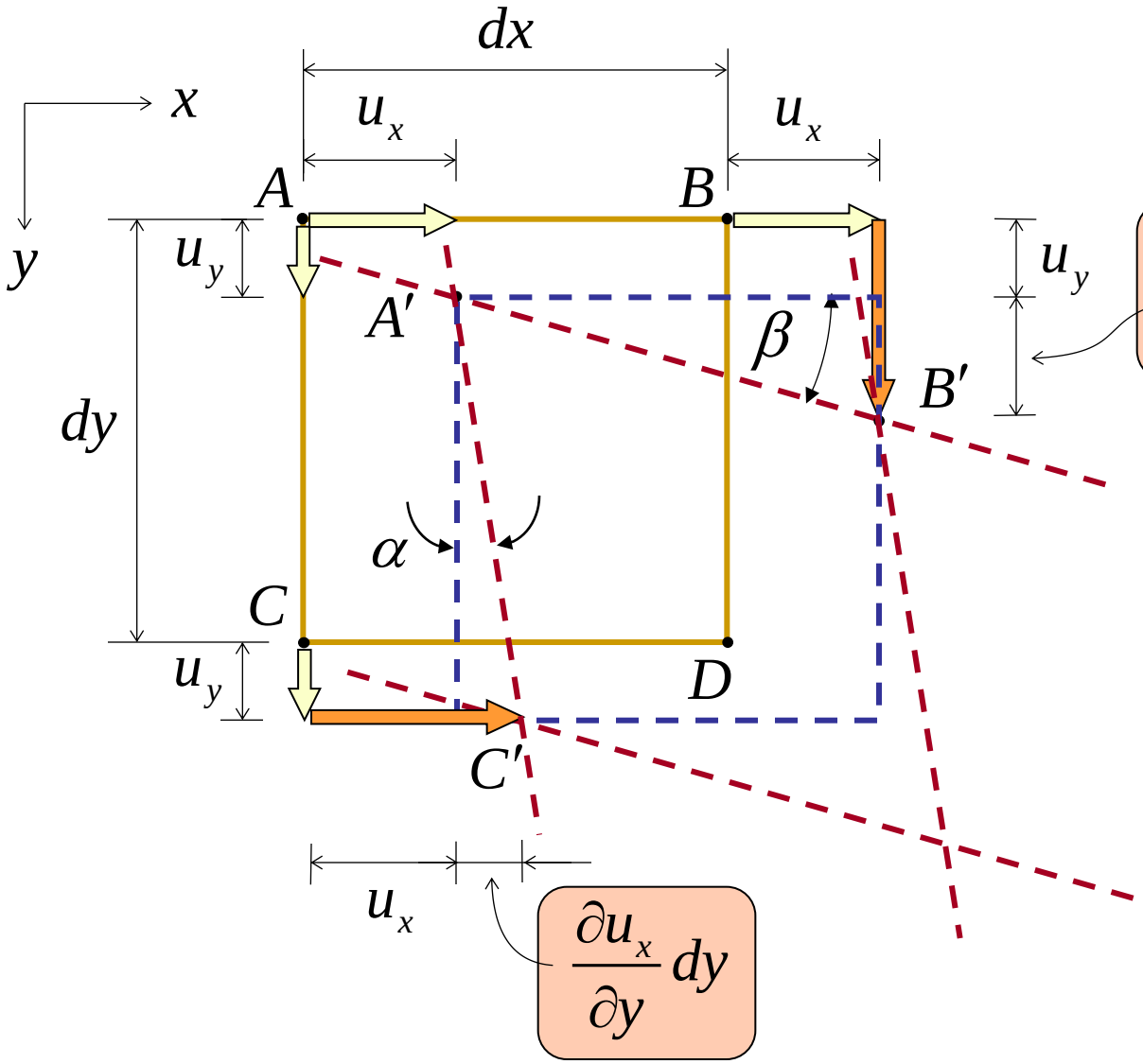
$$\beta = \frac{\frac{\partial u_y}{\partial x} dx}{dx} = \frac{\partial u_y}{\partial x}$$

$$\alpha = \frac{\frac{\partial u_x}{\partial y} dy}{dy} = \frac{\partial u_x}{\partial y}$$

$$\gamma_{xy} = \alpha + \beta$$

A plate loaded in-plane

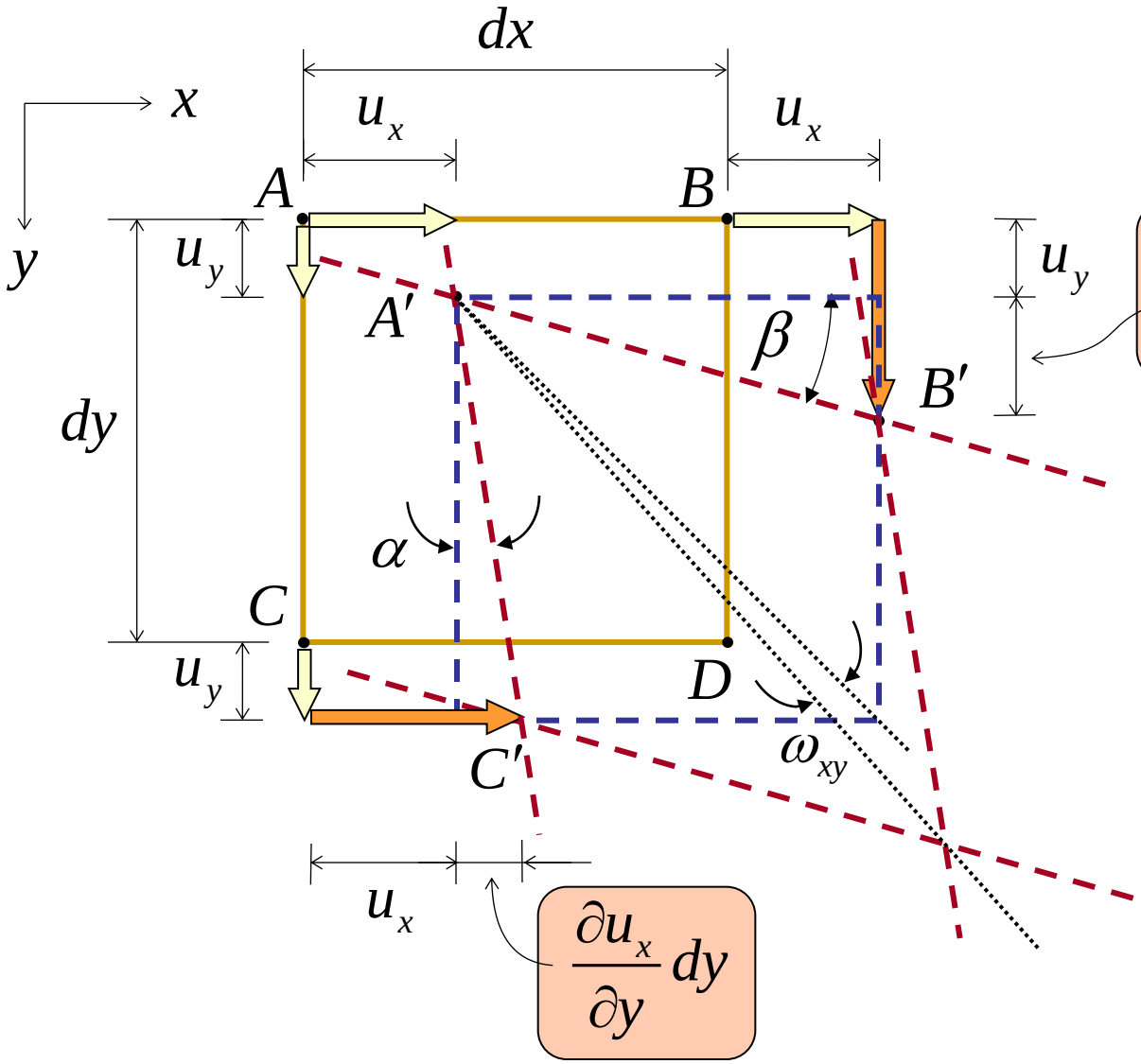
- shear strain



$$\gamma_{xy} = \alpha + \beta$$

A plate loaded in-plane

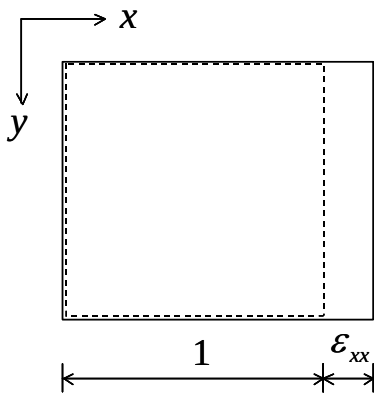
- shear rotation

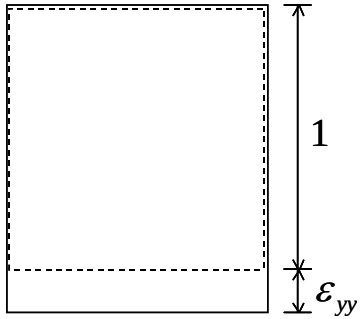


$$\gamma_{xy} = \alpha + \beta$$

$$\omega_{xy} = \frac{1}{2}(\beta - \alpha)$$

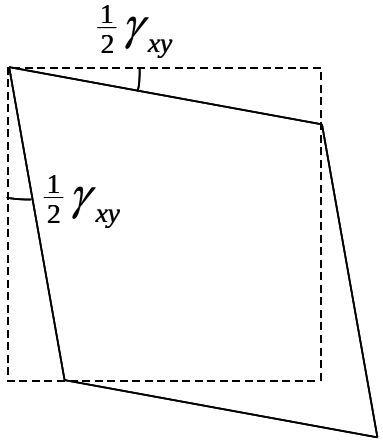
Summary of kinematic equations:





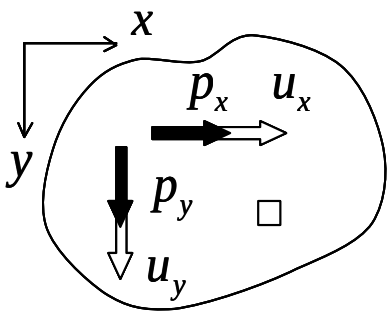
$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

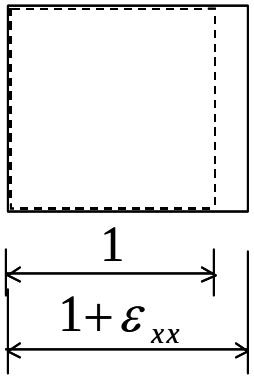
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

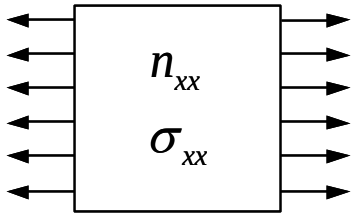


$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

A plate loaded in-plane







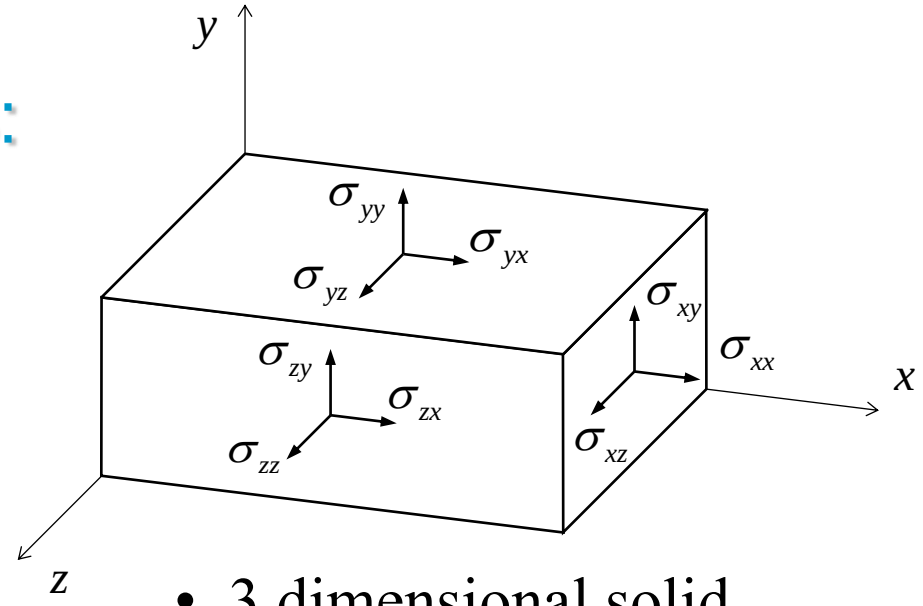
Derive:

$$\boldsymbol{u}=\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \boldsymbol{n}=\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p}=\begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

kinematic equations — *constitutive equations* — *equilibrium equations*

Constitutive equations:

$$\boldsymbol{e}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} \qquad \boldsymbol{n}=\begin{Bmatrix}n_{xx}\\ n_{yy}\\ n_{xy}\end{Bmatrix}$$



- 3 dimensional solid

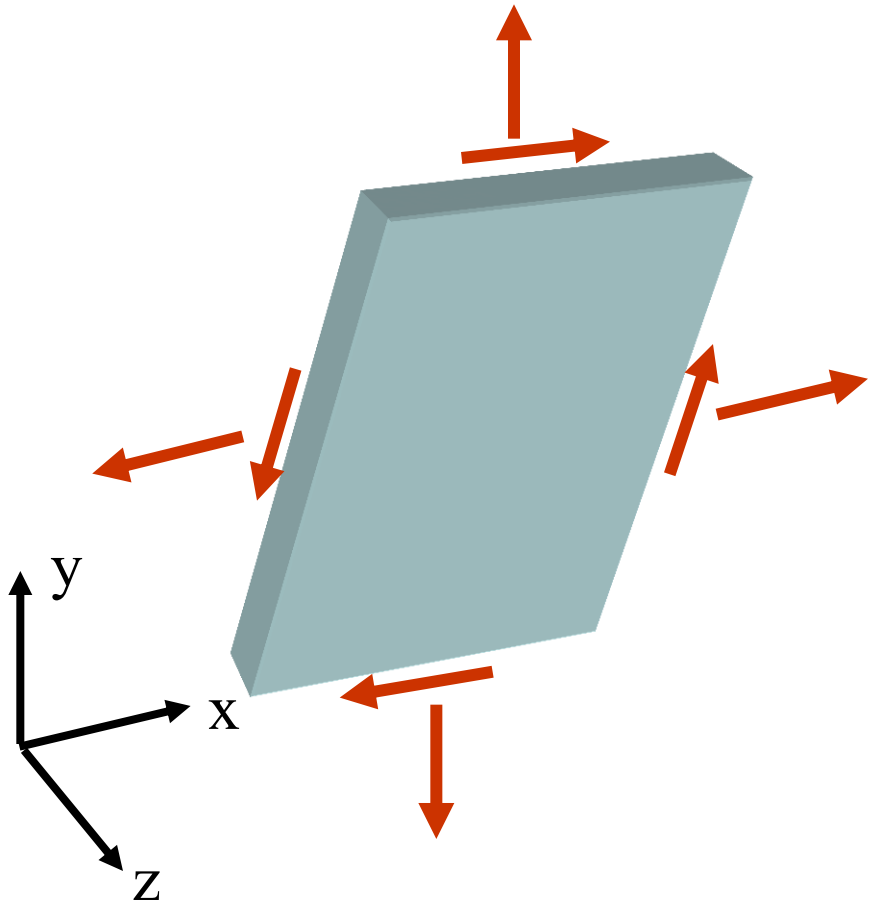
$$\varepsilon_{xx}=\frac{1}{E}\sigma_{xx}-\frac{\upsilon}{E}\left(\sigma_{yy}+\sigma_{zz}\right) \quad ; \quad \gamma_{xy}=\frac{2\left(1+\upsilon\right)}{E}\sigma_{xy}$$

$$\varepsilon_{yy}=\frac{1}{E}\sigma_{yy}-\frac{\upsilon}{E}\left(\sigma_{zz}+\sigma_{xx}\right) \quad ; \quad \gamma_{yz}=\frac{2\left(1+\upsilon\right)}{E}\sigma_{yz}$$

$$\varepsilon_{zz}=\frac{1}{E}\sigma_{zz}-\frac{\upsilon}{E}\left(\sigma_{xx}+\sigma_{yy}\right) \quad ; \quad \gamma_{zx}=\frac{2\left(1+\upsilon\right)}{E}\sigma_{zx}$$

Constitutive equations:

- plane stress solid



$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} - \frac{\upsilon}{E}\left(\sigma_{yy} + \cancel{\sigma_{zz}}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\upsilon}{E}\left(\cancel{\sigma_{zz}} + \sigma_{xx}\right)$$

$$\varepsilon_{zz} = \cancel{\frac{1}{E}\sigma_{zz}} - \frac{\upsilon}{E}\left(\sigma_{xx} + \sigma_{yy}\right)$$

$$\gamma_{xy} = \frac{2(1+\upsilon)}{E}\sigma_{xy}$$

$$\gamma_{yz} = \frac{2(1+\upsilon)}{E}\cancel{\sigma_{yz}}$$

$$\gamma_{zx} = \frac{2(1+\upsilon)}{E}\cancel{\sigma_{zx}}$$

Constitutive equations:

- plane stress solid

flexibility formulation:

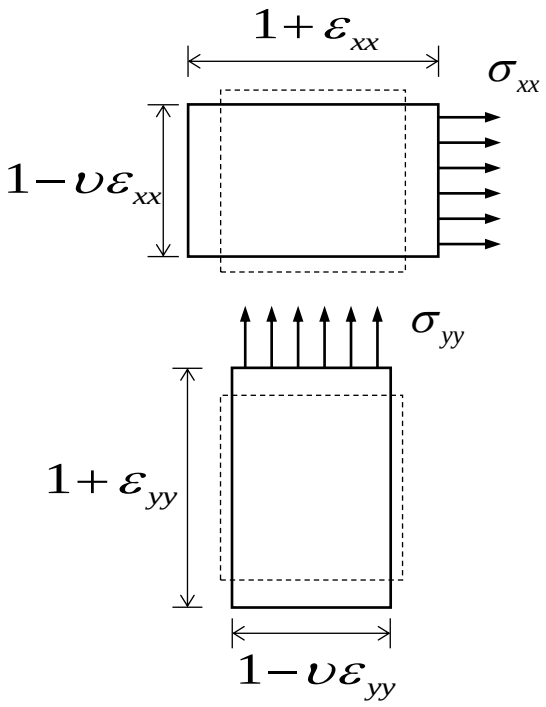
$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{Et} \begin{bmatrix} 1 & -\upsilon & 0 \\ -\upsilon & 1 & 0 \\ 0 & 0 & 2(1+\upsilon) \end{bmatrix} \begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \rightarrow \boxed{\boldsymbol{e} = \boldsymbol{C}\,\boldsymbol{n}}$$

stifness formulation:

$$\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} = \frac{Et}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \rightarrow \boxed{\boldsymbol{n} = \boldsymbol{D}\,\boldsymbol{e}}$$

Summary of constitutive equations:

Material with lateral contraction



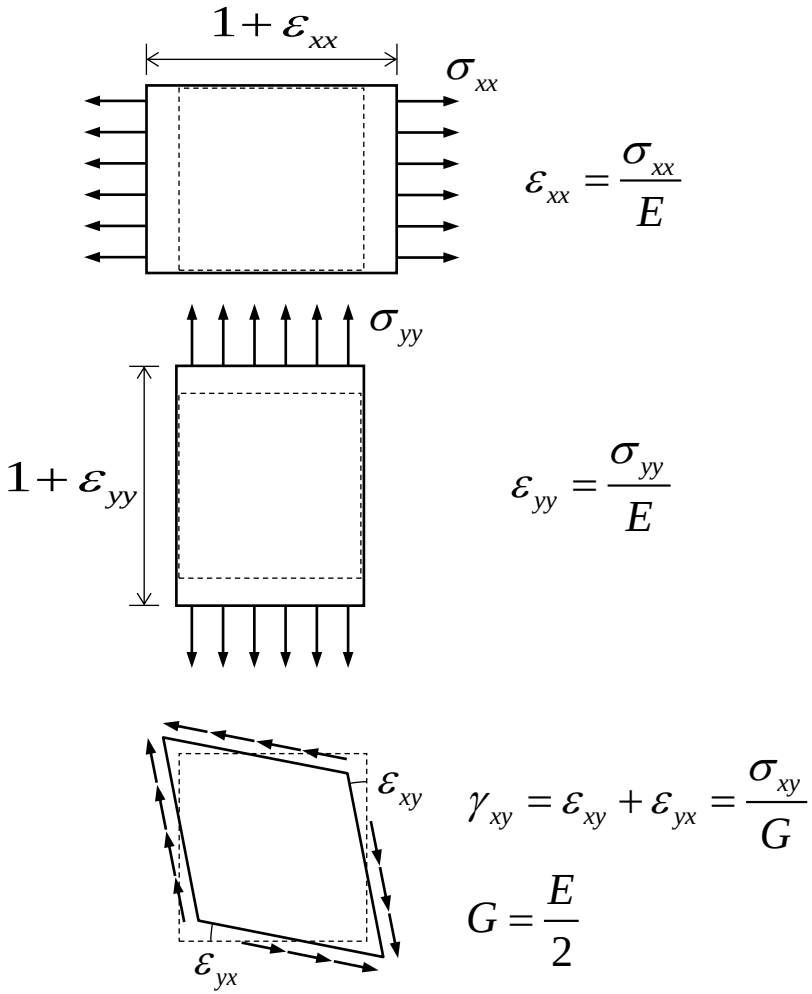
$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E}$$

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{\sigma_{xy}}{G}$$

$$G = \frac{E}{2(1+\upsilon)}$$

Material without lateral contraction



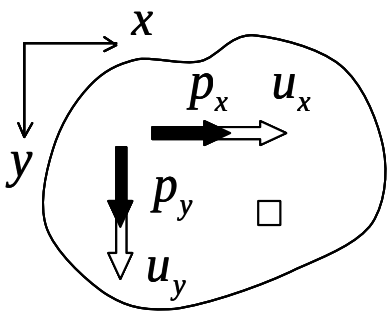
$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$$

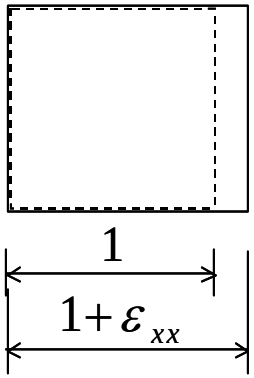
$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E}$$

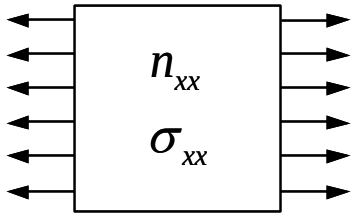
$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \frac{\sigma_{xy}}{G}$$

$$G = \frac{E}{2}$$

A plate loaded in-plane







Derive:

$$\boldsymbol{u}=\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \boldsymbol{n}=\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p}=\begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

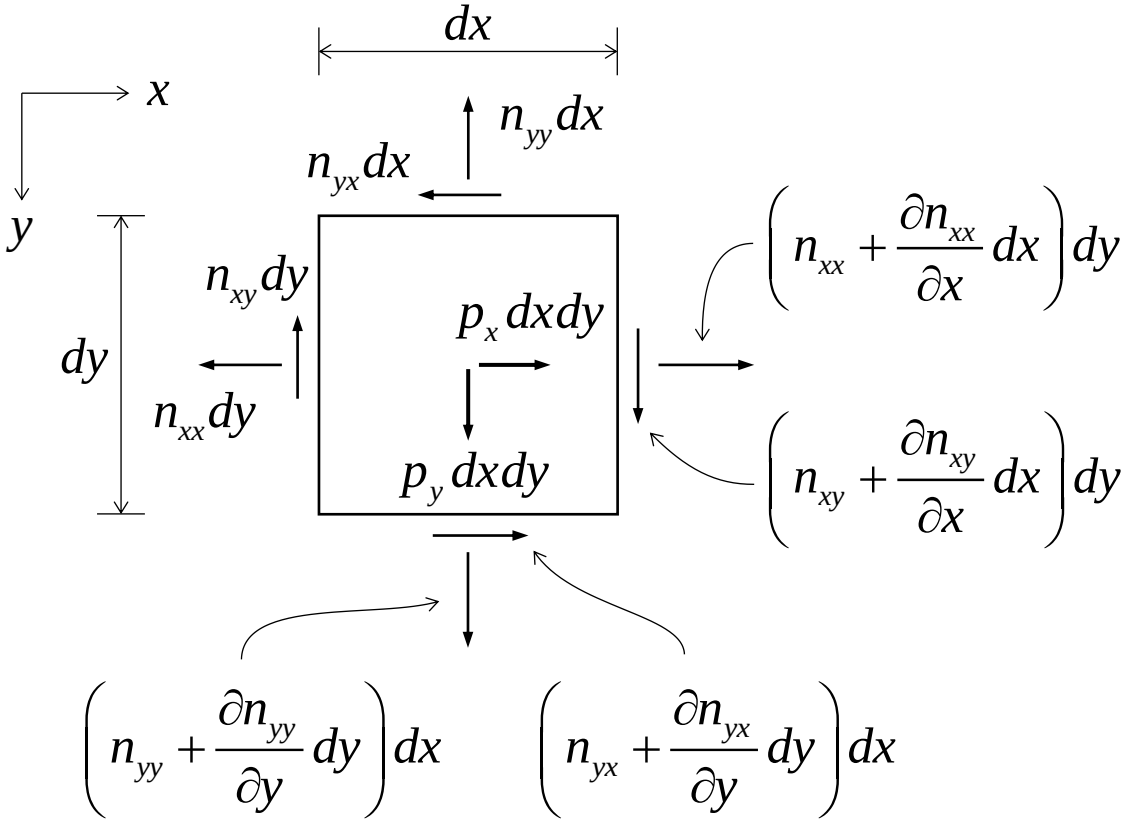
kinematic equations — *constitutive equations* — *equilibrium equations*

Equilibrium equations:

$$\boldsymbol{n} = \begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

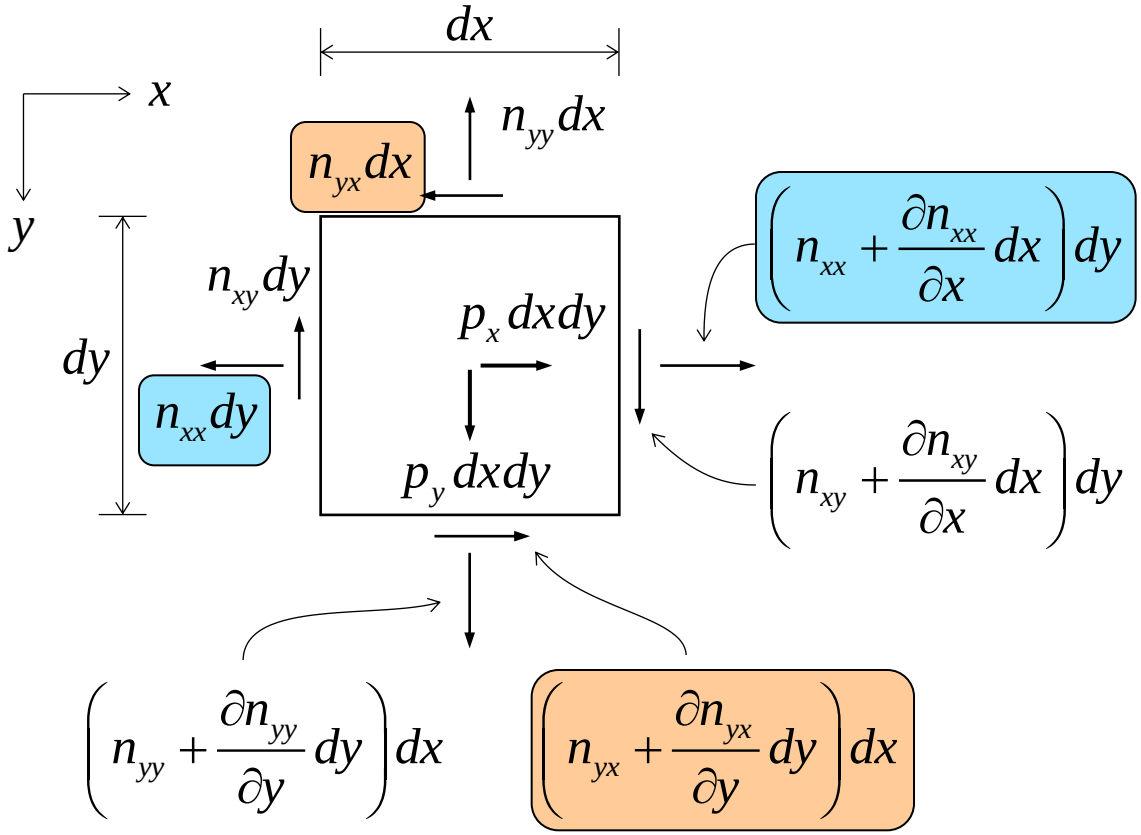
$$\sum n_x = 0$$

$$\sum n_y = 0$$



Equilibrium equations:

$$\sum n_x = 0$$



$$-n_{xx}dy+\left(n_{xx}+\frac{\partial n_{xx}}{\partial x}dx\right)dy-n_{yx}dx+\left(n_{yx}+\frac{\partial n_{yx}}{\partial y}dy\right)dx+p_x dxdy=0$$

Summary equilibrium equations:

$$\sum n_x = 0 \qquad -\left(\frac{\partial n_{xx}}{\partial x} + \frac{\partial n_{yx}}{\partial y}\right) = p_x$$

$$\sum n_y = 0 \qquad -\left(\frac{\partial n_{yy}}{\partial y} + \frac{\partial n_{xy}}{\partial x}\right) = p_y$$

$$\boldsymbol{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

kinematic equations — *constitutive equations* — *equilibrium equations*

Displacement Method

$$\boldsymbol{u}=\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \boldsymbol{n}=\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p}=\begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

kinematic equations — *constitutive equations* — *equilibrium equations*

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\ \gamma_{xy} &= \frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x} \end{aligned}$$

$$\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} = \frac{Et}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & (1-\upsilon)/2 \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

Displacement Method

$$\boldsymbol{u}=\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \boldsymbol{n}=\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p}=\begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$



kinematic equations

constitutive equations

equilibrium equations

$$n_{xx}=\frac{Et}{1-\upsilon^2}\left(\frac{\partial u_x}{\partial x}+\upsilon\frac{\partial u_y}{\partial y}\right)$$

$$n_{yy}=\frac{Et}{1-\upsilon^2}\left(\frac{\partial u_y}{\partial y}+\upsilon\frac{\partial u_x}{\partial x}\right)$$

$$n_{xy}=\frac{Et}{2(1+\upsilon)}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)$$

$$-\left(\frac{\partial n_{xx}}{\partial x}+\frac{\partial n_{yx}}{\partial y}\right)=p_x$$

$$-\left(\frac{\partial n_{yy}}{\partial y}+\frac{\partial n_{xy}}{\partial x}\right)=p_y$$

Displacement Method

$$-\frac{E\,t}{1-\upsilon^2}\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{1-\upsilon}{2}\frac{\partial^2 u_x}{\partial y^2}+\frac{1+\upsilon}{2}\frac{\partial^2 u_y}{\partial x\partial y}\right)=p_x$$

$$-\frac{E\,t}{1-\upsilon^2}\left(\frac{\partial^2 u_y}{\partial y^2}+\frac{1-\upsilon}{2}\frac{\partial^2 u_y}{\partial x^2}+\frac{1+\upsilon}{2}\frac{\partial^2 u_x}{\partial x\partial y}\right)=p_y$$

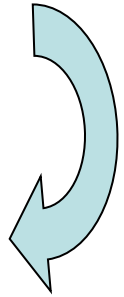
- in matrix differential operator:

$$-\frac{E\,t}{1-\upsilon^2}\begin{bmatrix}\dfrac{\partial^2}{\partial x^2}+\dfrac{1-\upsilon}{2}\dfrac{\partial^2}{\partial y^2} & \dfrac{1+\upsilon}{2}\dfrac{\partial^2}{\partial x\partial y}\\ \dfrac{1+\upsilon}{2}\dfrac{\partial^2}{\partial x\partial y} & \dfrac{\partial^2}{\partial y^2}+\dfrac{1-\upsilon}{2}\dfrac{\partial^2}{\partial x^2}\end{bmatrix}\begin{Bmatrix}u_x\\ u_y\end{Bmatrix}=\begin{Bmatrix}p_x\\ p_y\end{Bmatrix}$$

Displacement Method

$$-\frac{Et}{1-\upsilon^2}\left(\left(\frac{\partial^2}{\partial x^2}+\frac{1-\upsilon}{2}\frac{\partial^2}{\partial y^2}\right)u_x+\left(\frac{1+\upsilon}{2}\frac{\partial^2}{\partial x\partial y}\right)u_y\right)=p_x$$

$$-\frac{Et}{1-\upsilon^2}\left(\left(\frac{\partial^2}{\partial y^2}+\frac{1-\upsilon}{2}\frac{\partial^2}{\partial x^2}\right)u_y+\left(\frac{1+\upsilon}{2}\frac{\partial^2}{\partial x\partial y}\right)u_x\right)=p_y$$

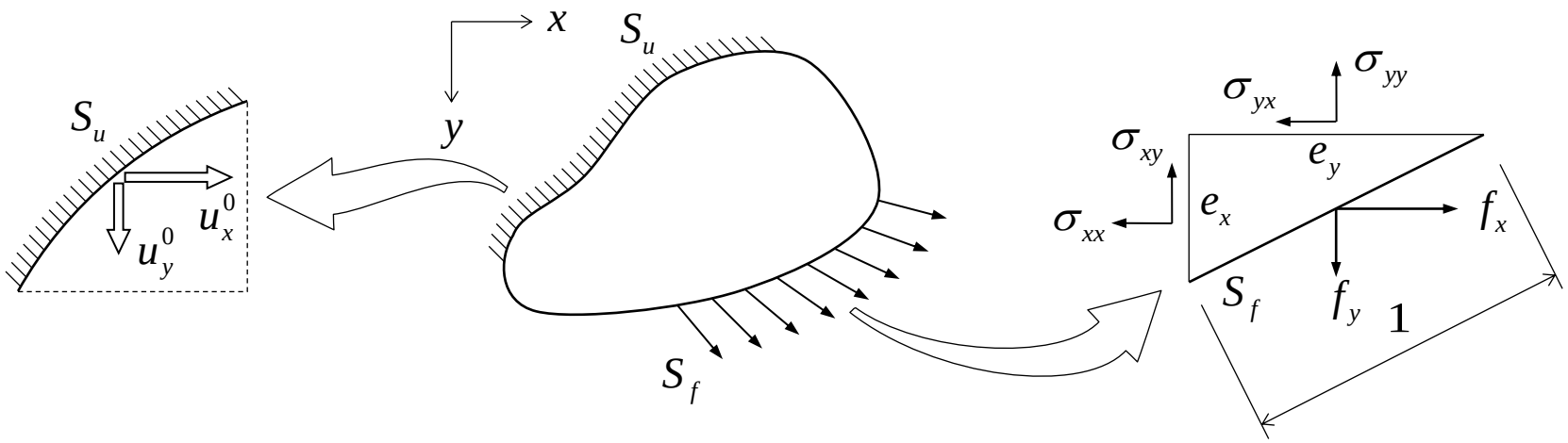


Biharmonic equation:

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

$$-\frac{Et}{2(1+\upsilon)}\nabla^2\nabla^2 u_x=\left(\frac{\partial^2}{\partial y^2}+\frac{1-\upsilon}{2}\frac{\partial^2}{\partial x^2}\right)p_x-\left(\frac{1+\upsilon}{2}\frac{\partial^2}{\partial x\partial y}\right)p_y$$

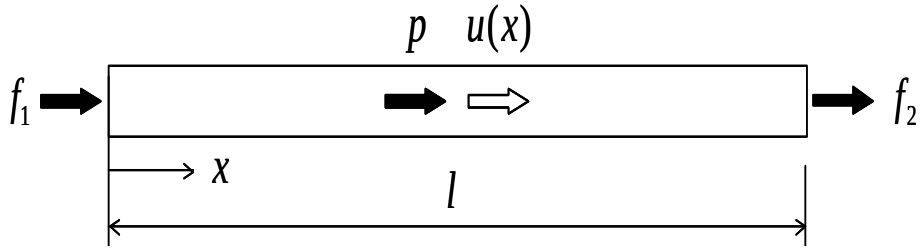
Boundary conditions



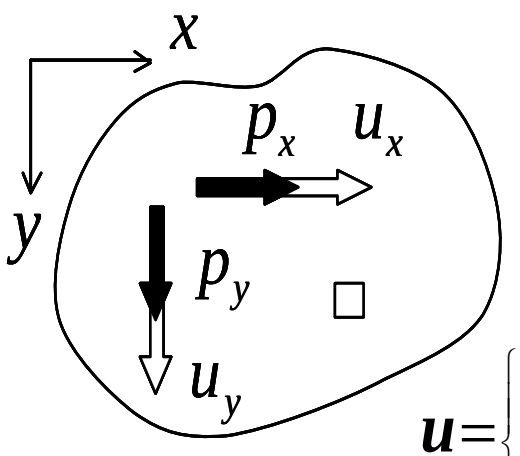
$$\left.\begin{aligned} u_x &= u_x^0 \\ u_y &= u_y^0 \end{aligned}\right\} \quad \text{on} \quad S_u$$

$$\left.\begin{aligned} \sigma_{xx}e_x + \sigma_{yx}e_y &= f_x \\ \sigma_{xy}e_x + \sigma_{yy}e_y &= f_y \end{aligned}\right\} \quad \text{on} \quad S_f$$

Truss vs. Plate



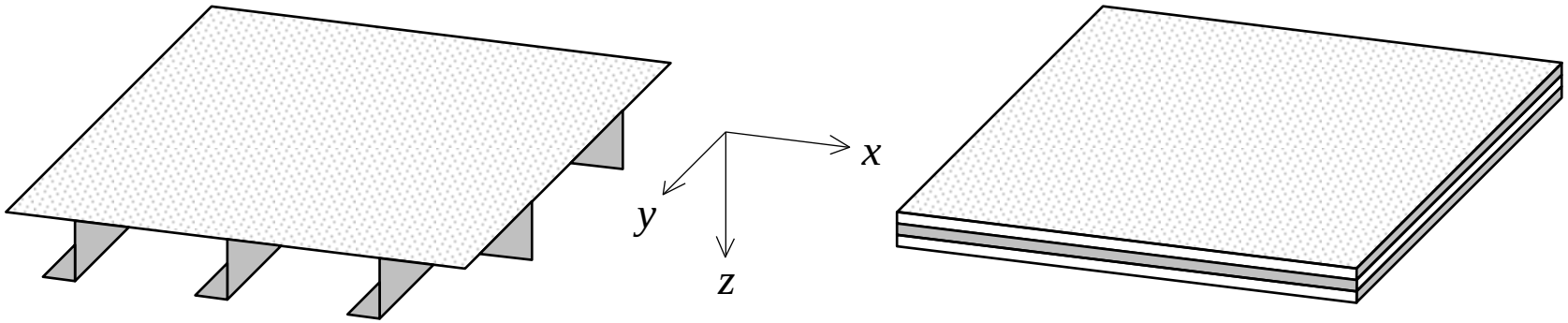
u ε N p



kinematic equations | *constitutive equations* | *equilibrium equations*

$$\boldsymbol{u}=\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} \qquad \boldsymbol{e}=\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \qquad \boldsymbol{n}=\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} \qquad \boldsymbol{p}=\begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

Orthotropic plates:



$$\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} = \begin{bmatrix} D_{xx} & D_{\upsilon} & 0 \\ D_{\upsilon} & D_{yy} & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad \text{or} \quad \boldsymbol{n = De}$$

Message of Chapter 1:

•Differential equations have been derived in the framework of the displacement method (stiffness method). We distinguish the following four quantities: displacements, their associate external loading, deformations and their associate internal stress resultants.

•The plane stress state in the membrane plate theory is defined by two degrees of freedom (displacements) and two corresponding load components. The three deformations are strains, and the associate stress resultants are membrane forces.

Message of the Chapter 1 (cont.):

•Three basic sets of equations must be derived: kinematic, constitutive and equilibrium equations. The kinematic equations relate the deformations (strains) to the displacements. The constitutive equations define the relationship between deformations and stress resultants (membrane forces). The equilibrium relations relate the stress resultants to the external loads.

•Subsequent substitution of the one in the other leads to the required differential equations, which are equilibrium equations, expressed in displacements. In the membrane plate theory we obtain two simultaneous partial differential equations of the second-order, which can be replaced by one fourth-order bi-harmonic equation.

Message of the Chapter 1 (cont.):

•In the plate theory two boundary conditions can be specified per plate edge, either the displacement normal to the edge (kinematic condition) or the edge load in that direction (dynamic condition). In the same way either the displacement parallel to the edge or the load in that direction.