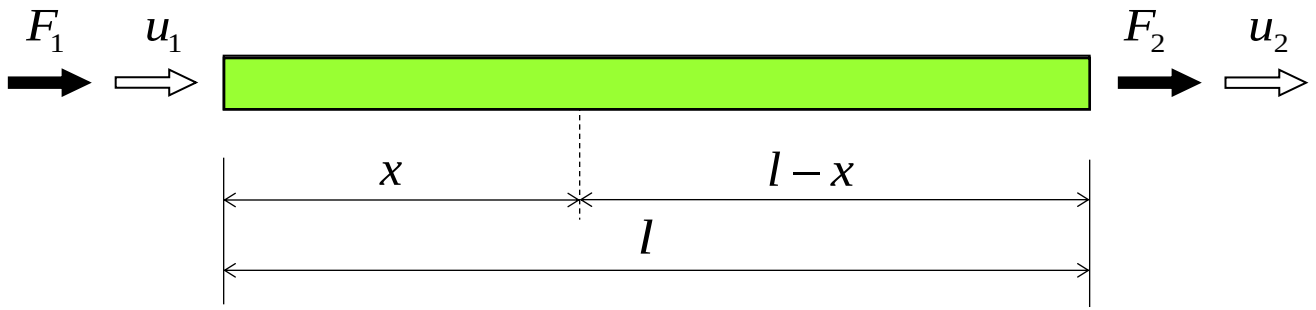
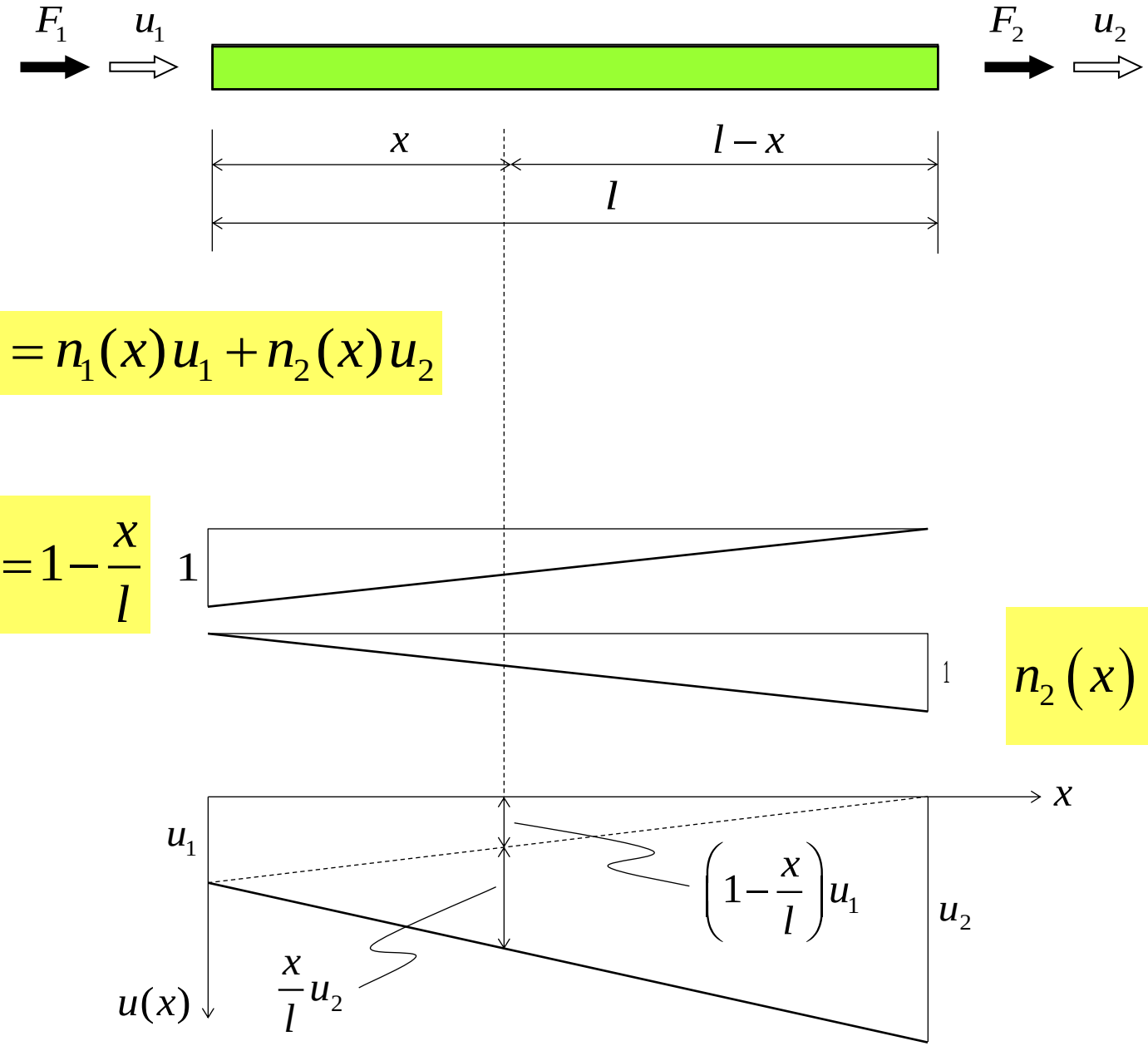
Finite Element Method (FEM):

- truss element



$$\boldsymbol{u} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad ; \quad \boldsymbol{f} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \Rightarrow \quad \boldsymbol{f = K\,u}$$


F_1
u_1
F_2
u_2
x
$l-x$
l
$u_c(x)=n_1(x)u_1+n_2(x)u_2$
$n_1(x)=1-\frac{x}{l}$
1
1
$n_2(x)=\frac{x}{l}$
x
u_1
$\left(1-\frac{x}{l}\right)u_1$
u_2
$u(x)$
$\frac{x}{l}u_2$

$$u_c(x) = n_1(x)u_1 + n_2(x)u_2$$

$$= \begin{bmatrix} n_1(x) & n_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\Rightarrow \quad u_c(x) = \boldsymbol{N}\,\boldsymbol{u}$$

$\boldsymbol{u}$ $\qquad\qquad$ $\boldsymbol{\varepsilon}$

kinematic equations

$$\varepsilon = \frac{du}{dx}$$

$$= \begin{bmatrix} b_1(x) & b_2(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$b_1(x) = \frac{dn_1}{dx} \quad ; \quad b_2(x) = \frac{dn_2}{dx}$$

$$\Rightarrow \quad \boldsymbol{\varepsilon} = \boldsymbol{B}\,\boldsymbol{u}$$

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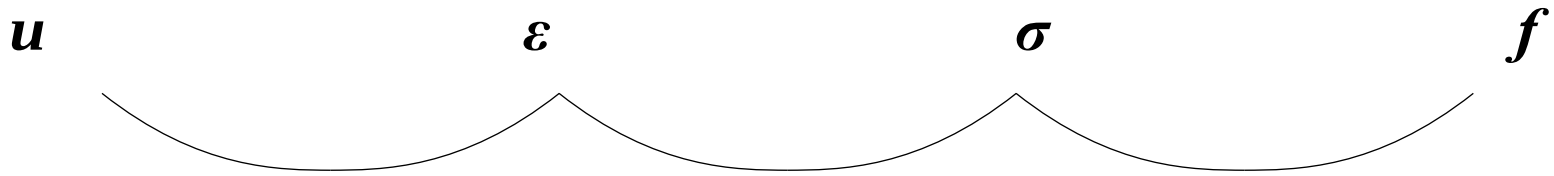


$$\boldsymbol{\varepsilon} = \boldsymbol{B}\,\boldsymbol{u} \qquad \boldsymbol{\sigma} = \boldsymbol{E}\,\boldsymbol{\varepsilon} \qquad ?$$

$$W_{external} = W_{internal}$$

$$\delta\boldsymbol{u}^T\,\boldsymbol{f} = A\int_0^l \delta\boldsymbol{\varepsilon}^T\,\boldsymbol{\sigma}\,dx$$

$$\delta\boldsymbol{\varepsilon} = \boldsymbol{B}\,\delta\boldsymbol{u}$$

$$= \delta\boldsymbol{u}^T A\int_0^l \boldsymbol{B}^T\boldsymbol{\sigma}\,dx$$

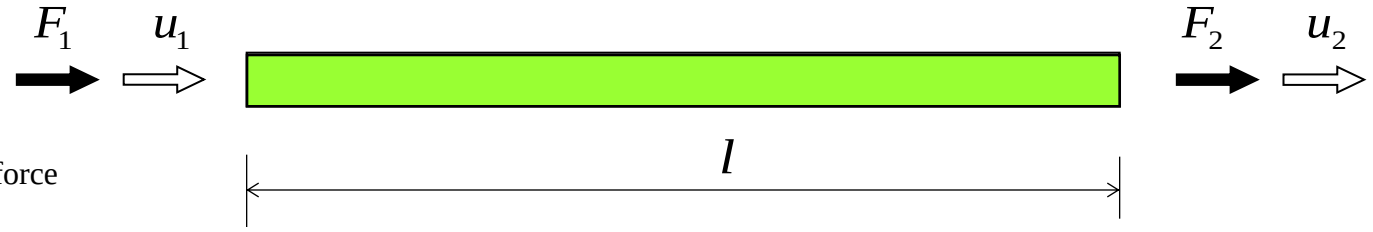
$$\boldsymbol{f} = A\int_0^l \boldsymbol{B}^T\boldsymbol{\sigma}\,dx$$

- Nodal forces due to element stresses

$$\boldsymbol{f} = A\int_0^l \boldsymbol{B}^T \boldsymbol{\sigma}\, dx$$

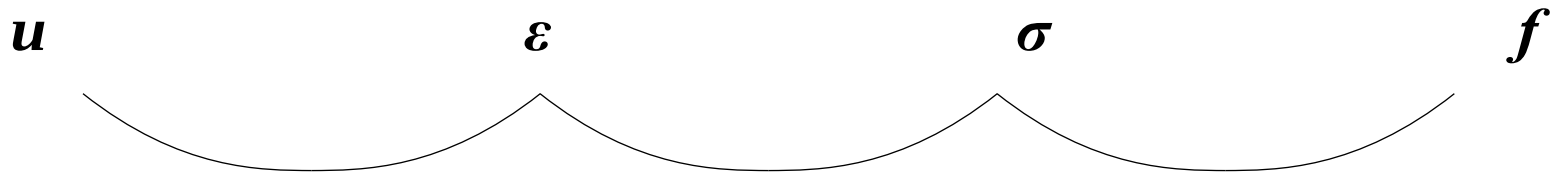
$$\boldsymbol{B} = \begin{bmatrix} \frac{dn_1}{dx} & \frac{dn_2}{dx} \end{bmatrix} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}$$

$$A\int_0^l \begin{bmatrix} -\frac{1}{l} \\ \\ \frac{1}{l} \end{bmatrix} [\,\sigma\,]\, dx = \begin{Bmatrix} -\int_0^l \frac{N}{l}\, dx \\ \\ \int_0^l \frac{N}{l}\, dx \end{Bmatrix} = \begin{Bmatrix} -N \\ \\ N \end{Bmatrix} = \begin{Bmatrix} F_1 \\ \\ F_2 \end{Bmatrix} = \boldsymbol{f}$$



Note : N stands for force
not shape function!

- Element stiffness matrix



$$\boldsymbol{\varepsilon} = \boldsymbol{B}\boldsymbol{u} \qquad \boldsymbol{\sigma} = E\boldsymbol{\varepsilon} \qquad \boldsymbol{f} = A\int_0^l \boldsymbol{B}^T\boldsymbol{\sigma}\,dx$$

$$\boldsymbol{f} = \left(\int_0^l \boldsymbol{B}^T\boldsymbol{D}\,\boldsymbol{B}\,dx\right)\boldsymbol{u} \quad \Rightarrow \quad \boxed{\boldsymbol{K} = \int_0^l \boldsymbol{B}^T\boldsymbol{D}\,\boldsymbol{B}\,dx}$$

$$\boldsymbol{D} = EA$$

$$\boldsymbol{K} = \int_0^l \boldsymbol{B}^T \boldsymbol{D}\, \boldsymbol{B}\, dx$$

$$\boldsymbol{B} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}$$

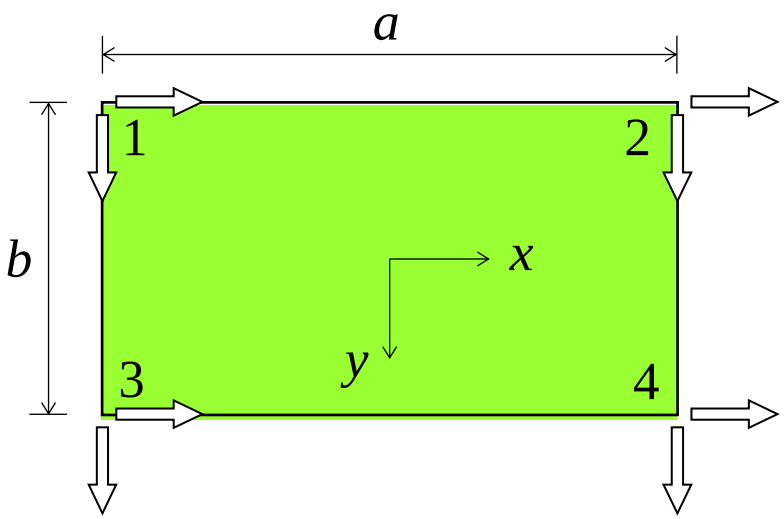
$$\boldsymbol{D} = EA$$

$$= \int_0^l \begin{bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{bmatrix} [EA] \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} dx$$

$$= \begin{bmatrix} \int_0^l \frac{EA}{l^2} dx & \int_0^l \frac{-EA}{l^2} dx \\ \int_0^l \frac{-EA}{l^2} dx & \int_0^l \frac{EA}{l^2} dx \end{bmatrix}$$

$$\Rightarrow \quad \boldsymbol{K} = \begin{bmatrix} \frac{EA}{l} & \frac{-EA}{l} \\ \frac{-EA}{l} & \frac{EA}{l} \end{bmatrix}$$

Rectangular Element



$$\boldsymbol{u}^T = \{u_{x1} \quad u_{y1} \quad u_{x2} \quad u_{y2} \quad u_{x3} \quad u_{y3} \quad u_{x4} \quad u_{y4}\}$$

$$\boldsymbol{f}^T = \{F_{x1} \quad F_{y1} \quad F_{x2} \quad F_{y2} \quad F_{x3} \quad F_{y3} \quad F_{x4} \quad F_{y4}\}$$

at nodes

$$\boldsymbol{\varepsilon}(x,y)^T = \{\varepsilon_{xx}(x,y) \quad \varepsilon_{yy}(x,y) \quad \gamma_{xy}(x,y)\}$$

$$\boldsymbol{\sigma}(x,y)^T = \{n_{xx}(x,y) \quad n_{yy}(x,y) \quad n_{xy}(x,y)\}$$

$$\boldsymbol{u_c}(x,y)^T = \{u_x(x,y) \quad u_y(x,y)\}$$

$$\boldsymbol{p}(x,y)^T = \{p_x(x,y) \quad p_y(x,y)\}$$

over element area

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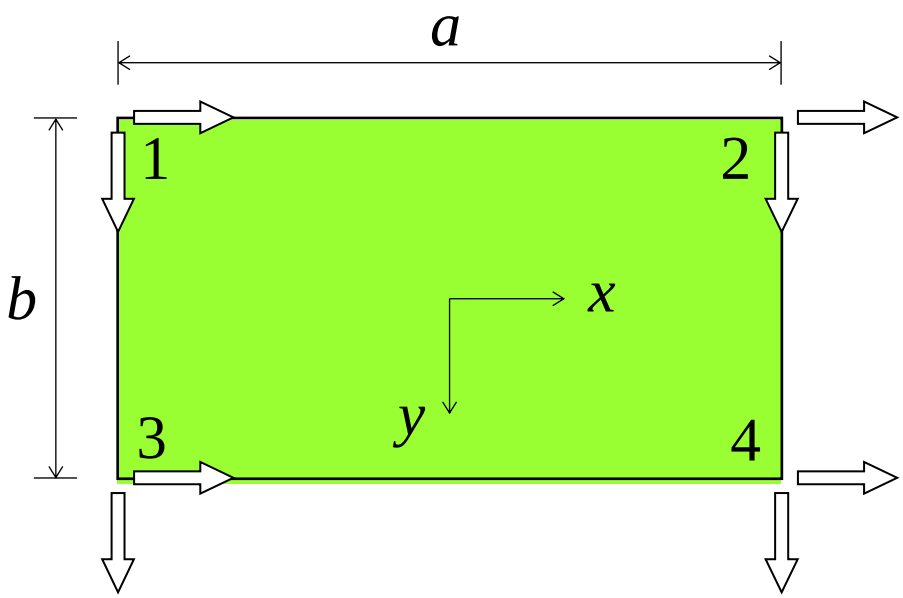
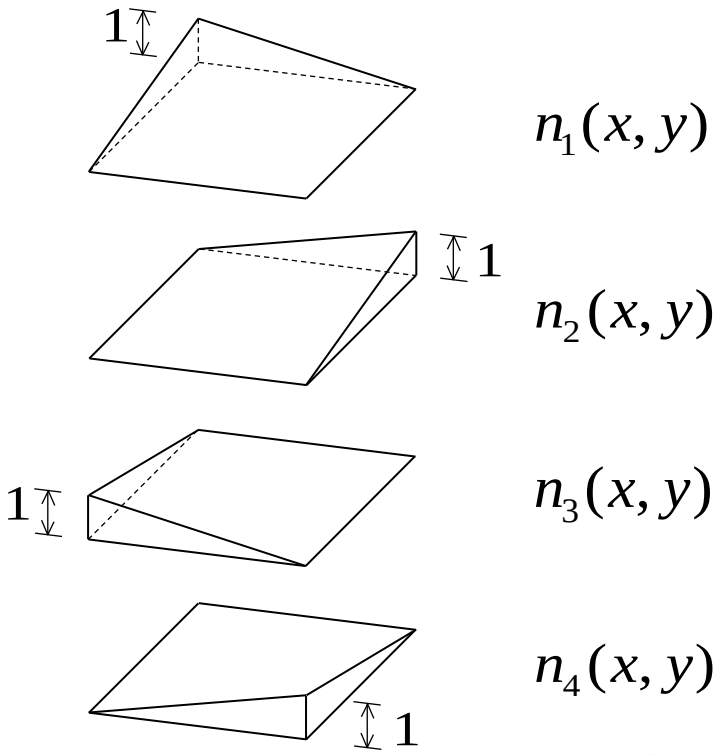
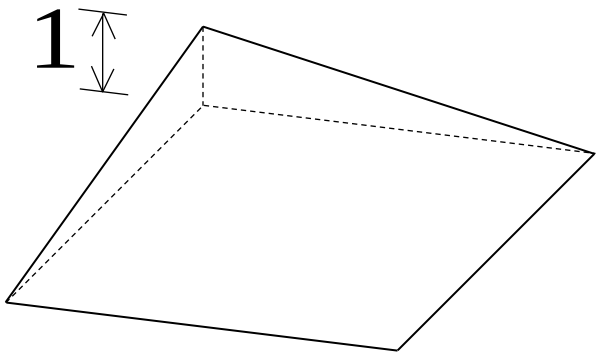
Rectangular Element

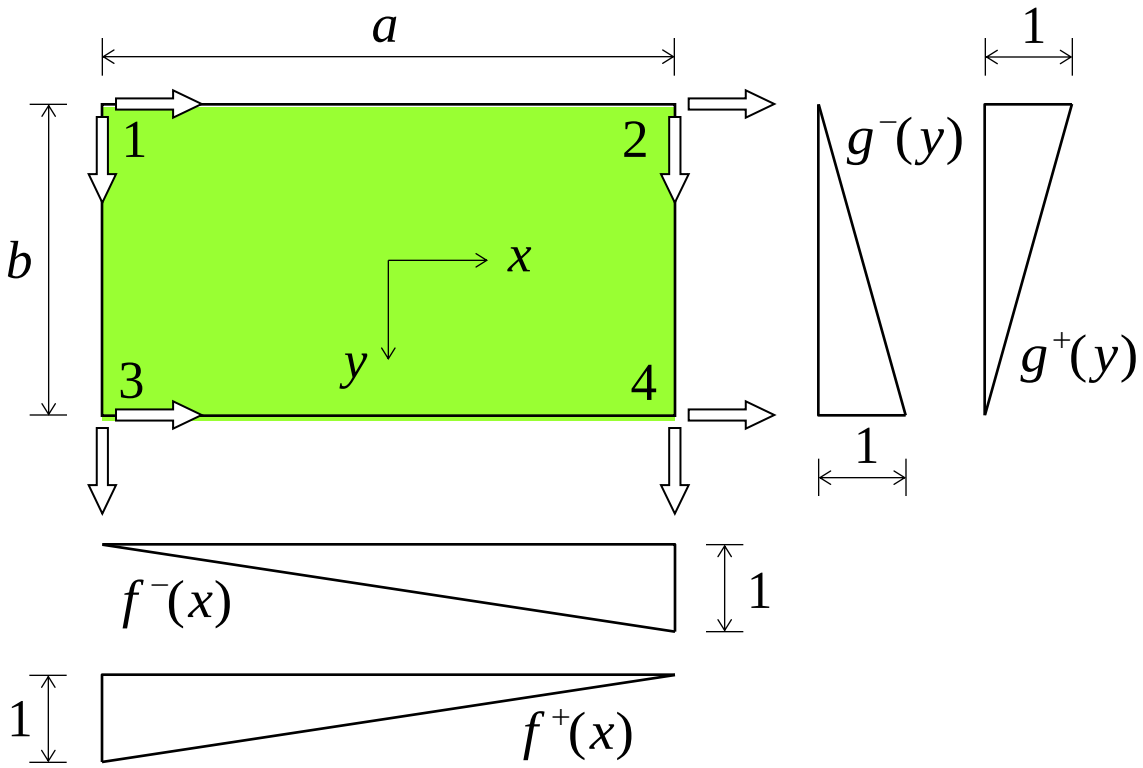
$$u_x(x,y) = n_1(x,y)u_{x1} + n_2(x,y)u_{x2} + n_3(x,y)u_{x3} + n_4(x,y)u_{x4}$$
$$u_y(x,y) = n_1(x,y)u_{y1} + n_2(x,y)u_{y2} + n_3(x,y)u_{y3} + n_4(x,y)u_{y4}$$





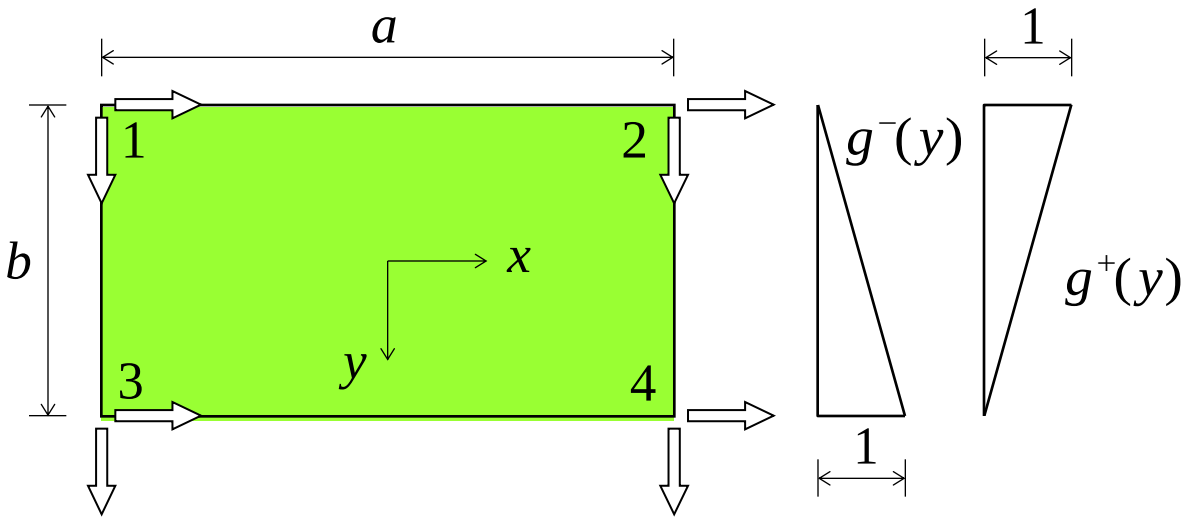
$$n_1(x,y) = ?$$

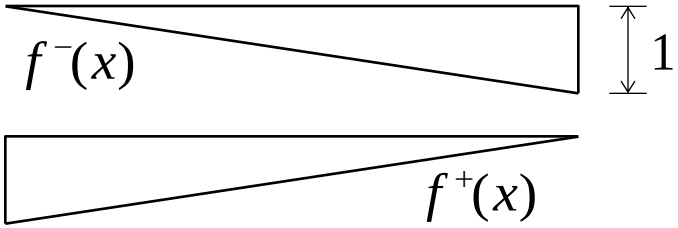




$$f^-(x) = \frac{1}{2} + \frac{x}{a} \quad ; \quad g^-(y) = \frac{1}{2} + \frac{y}{b}$$

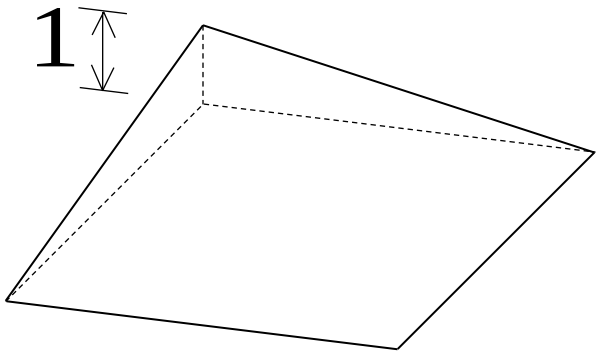
$$f^+(x) = \frac{1}{2} - \frac{x}{a} \quad ; \quad g^+(y) = \frac{1}{2} - \frac{y}{b}$$





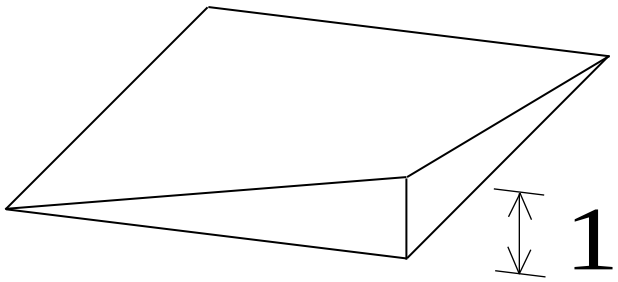
$$f^-(x) = \frac{1}{2} + \frac{x}{a} \quad ; \quad g^-(y) = \frac{1}{2} + \frac{y}{b}$$

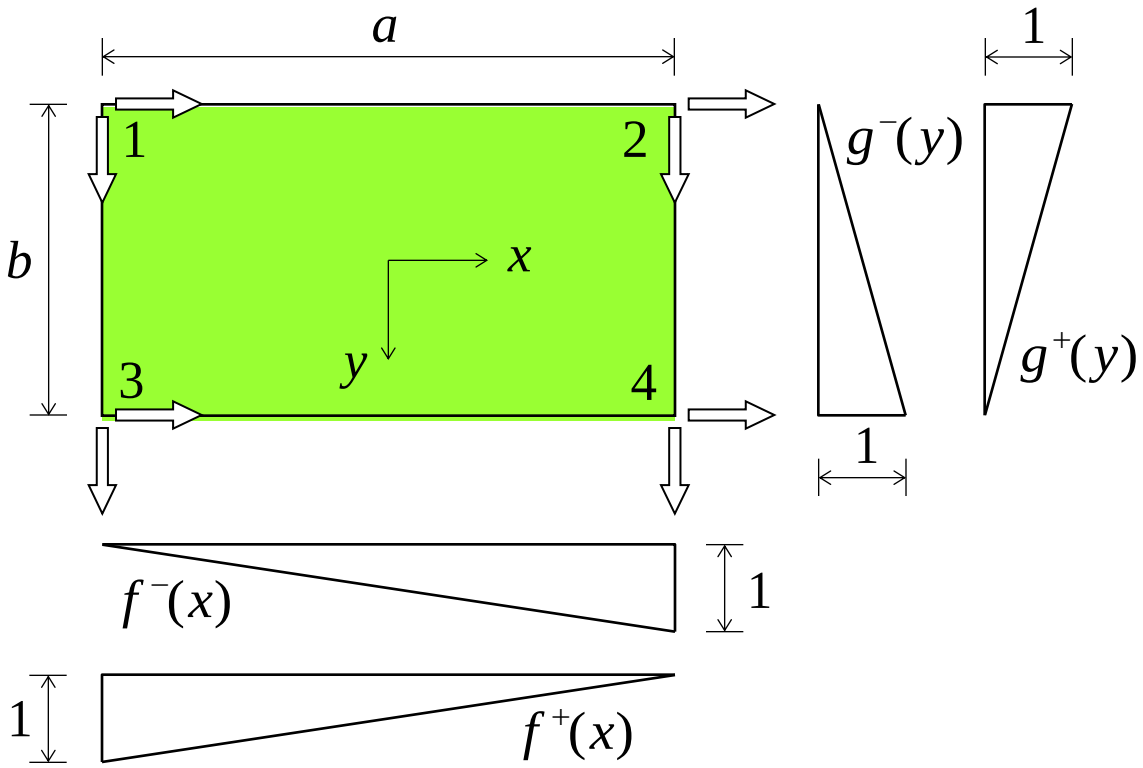
$$f^+(x) = \frac{1}{2} - \frac{x}{a} \quad ; \quad g^+(y) = \frac{1}{2} - \frac{y}{b}$$



$$n_1(x,y) = f^+(x) \cdot g^+(y)$$

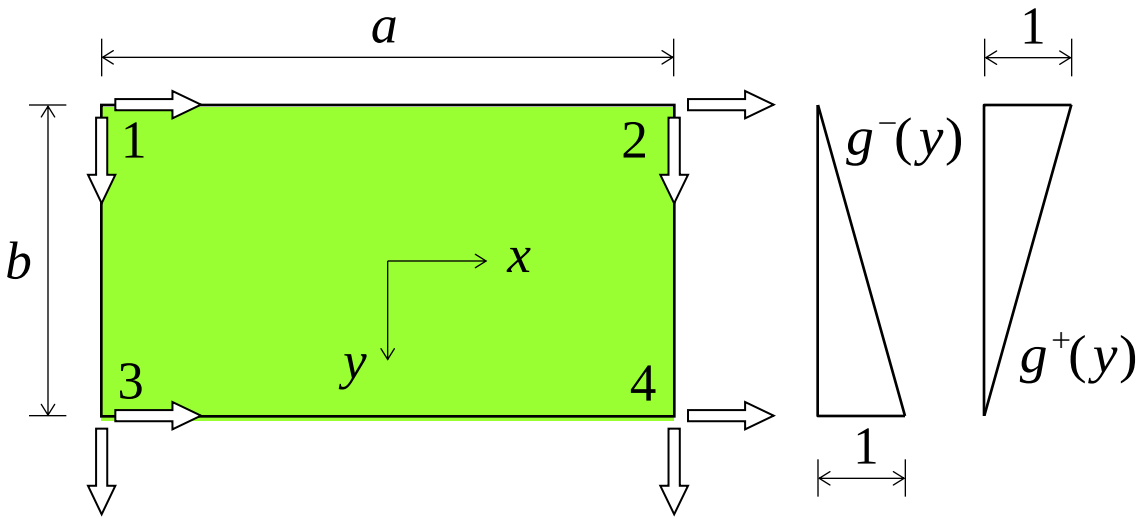
$$n_4(x,y) = ?$$





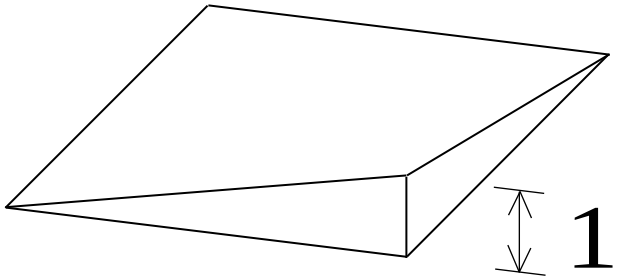
$$f^-(x) = \frac{1}{2} + \frac{x}{a} \quad ; \quad g^-(y) = \frac{1}{2} + \frac{y}{b}$$

$$f^+(x) = \frac{1}{2} - \frac{x}{a} \quad ; \quad g^+(y) = \frac{1}{2} - \frac{y}{b}$$

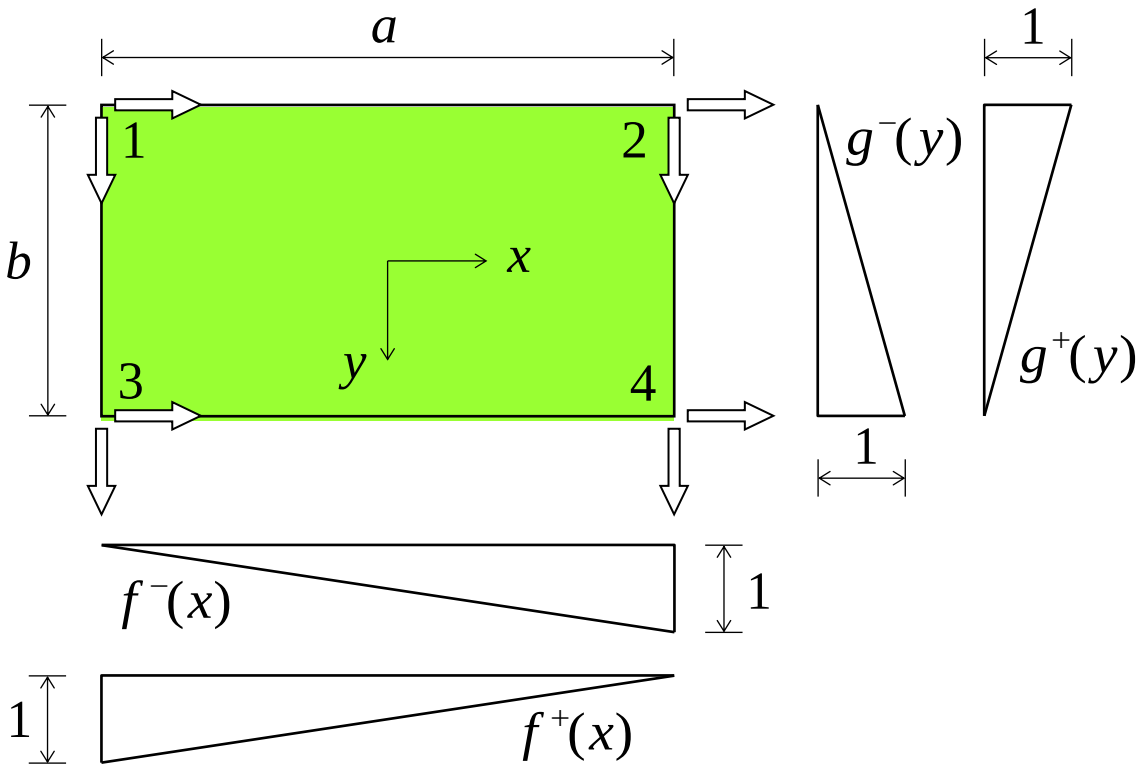


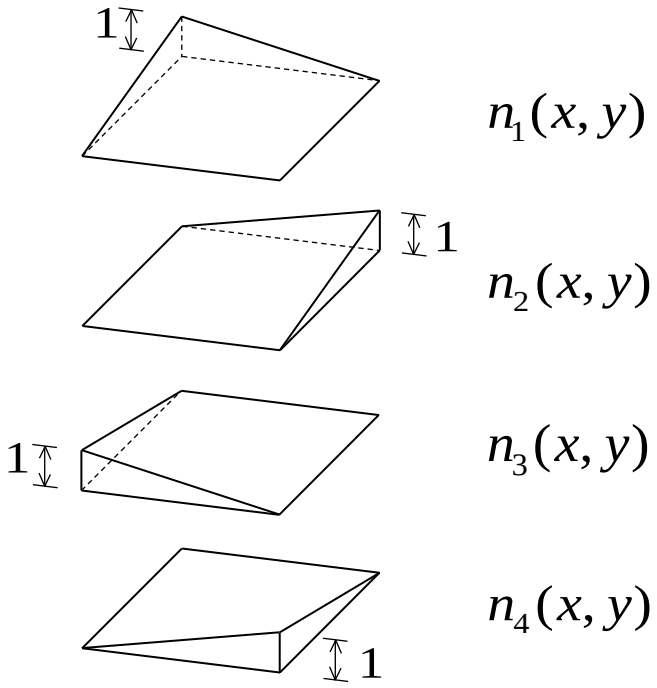
$$f^-(x) = \frac{1}{2} + \frac{x}{a} \quad ; \quad g^-(y) = \frac{1}{2} + \frac{y}{b}$$

$$f^+(x) = \frac{1}{2} - \frac{x}{a} \quad ; \quad g^+(y) = \frac{1}{2} - \frac{y}{b}$$



$$n_4(x,y) = f^-(x) \cdot g^-(y)$$





$$f^-(x) = \frac{1}{2} + \frac{x}{a} \quad ; \quad g^-(y) = \frac{1}{2} + \frac{y}{b}$$

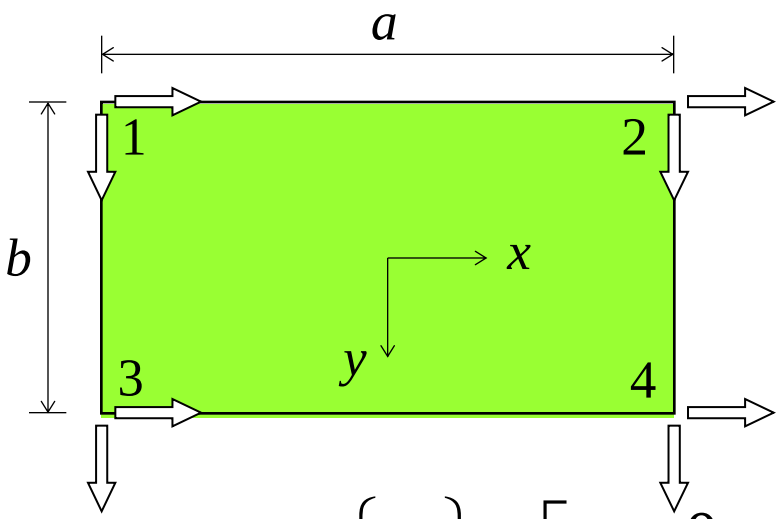
$$f^+(x) = \frac{1}{2} - \frac{x}{a} \quad ; \quad g^+(y) = \frac{1}{2} - \frac{y}{b}$$

$$n_1(x,y) = f^+(x) \cdot g^+(y)$$

$$n_2(x,y) = f^-(x) \cdot g^+(y)$$

$$n_3(x,y) = f^+(x) \cdot g^-(y)$$

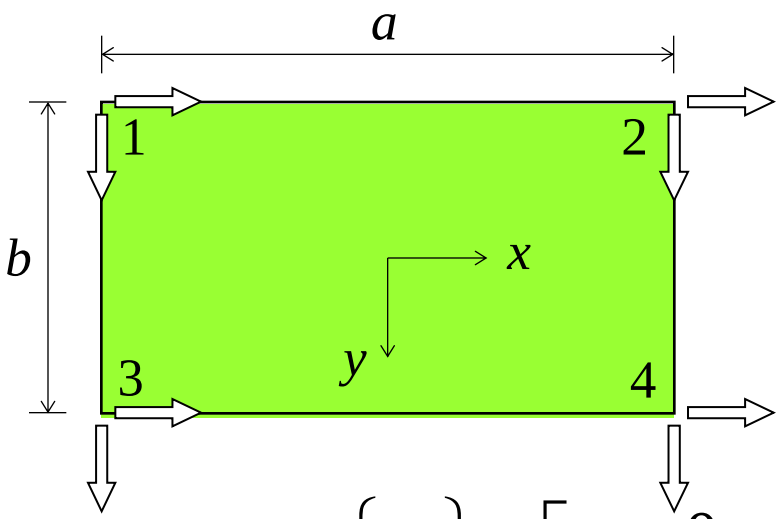
$$n_4(x,y) = f^-(x) \cdot g^-(y)$$



$$n_1(x,y) = f^+(x)\cdot g^+(y)$$
$$n_2(x,y) = f^-(x)\cdot g^+(y)$$
$$n_3(x,y) = f^+(x)\cdot g^-(y)$$
$$n_4(x,y) = f^-(x)\cdot g^-(y)$$

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} n_1 & 0 & n_2 & 0 & n_3 & 0 & n_4 & 0 \\ 0 & n_1 & 0 & n_2 & 0 & n_3 & 0 & n_4 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

$$\boldsymbol{u_c} = \boldsymbol{N}\,\boldsymbol{u}$$



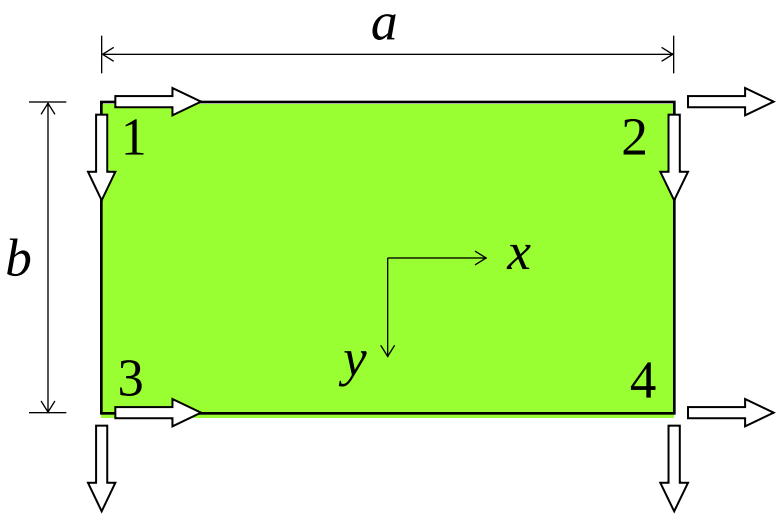
$$n_1(x,y) = f^+(x)\cdot g^+(y)$$
$$n_2(x,y) = f^-(x)\cdot g^+(y)$$
$$n_3(x,y) = f^+(x)\cdot g^-(y)$$
$$n_4(x,y) = f^-(x)\cdot g^-(y)$$

$$\boldsymbol{u_c} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} n_1 & 0 & n_2 & 0 & n_3 & 0 & n_4 & 0 \\ 0 & n_1 & 0 & n_2 & 0 & n_3 & 0 & n_4 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

$$\boldsymbol{u_c} = \boldsymbol{N}\,\boldsymbol{u}$$

$\boldsymbol{\varepsilon}$ $\qquad$ $\boldsymbol{u}$

kinematic equations ?



$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\, \boldsymbol{u}_c$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = ?$$

$$u_x = n_1 u_{x1} + n_2 u_{x2} + n_3 u_{x3} + n_4 u_{x4}$$

$$n_1(x,y) = f^+(x) \cdot g^+(y)$$
$$n_2(x,y) = f^-(x) \cdot g^+(y)$$
$$n_3(x,y) = f^+(x) \cdot g^-(y)$$
$$n_4(x,y) = f^-(x) \cdot g^-(y)$$

$$\frac{df^-}{dx} = \frac{1}{a} \quad ; \quad \frac{dg^-}{dy} = \frac{1}{b}$$

$$\frac{df^+}{dx} = -\frac{1}{a} \quad ; \quad \frac{dg^+}{dy} = -\frac{1}{b}$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{g^+}{a} u_{x1} + \frac{g^+}{a} u_{x2} - \frac{g^-}{a} u_{x3} + \frac{g^-}{a} u_{x4}$$

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$$\boldsymbol{\varepsilon} = \boldsymbol{B}\,\boldsymbol{u}_c$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{g^+}{a}u_{x1} + \frac{g^+}{a}u_{x2} - \frac{g^-}{a}u_{x3} + \frac{g^-}{a}u_{x4}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = -\frac{f^+}{b}u_{y1} - \frac{f^-}{b}u_{y2} + \frac{f^+}{b}u_{y3} + \frac{f^-}{b}u_{y4}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\frac{f^+}{b}u_{x1} - \frac{f^-}{b}u_{x2} + \frac{f^+}{b}u_{x3} + \frac{f^-}{b}u_{x4} +$$

$$-\frac{g^+}{a}u_{y1} + \frac{g^+}{a}u_{y2} - \frac{g^-}{a}u_{y3} + \frac{g^-}{a}u_{y4}$$

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$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \left[\begin{array}{cccccccc} -g^+/a & 0 & g^+/a & 0 & -g^-/a & 0 & g^-/a & 0 \\ 0 & -f^+/b & 0 & -f^-/b & 0 & f^+/b & 0 & f^-/b \\ -f^+/b & -g^+/a & -f^-/b & g^+/a & f^+/b & -g^-/a & f^-/b & g^-/a \end{array}\right] \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\,\boldsymbol{u}$$

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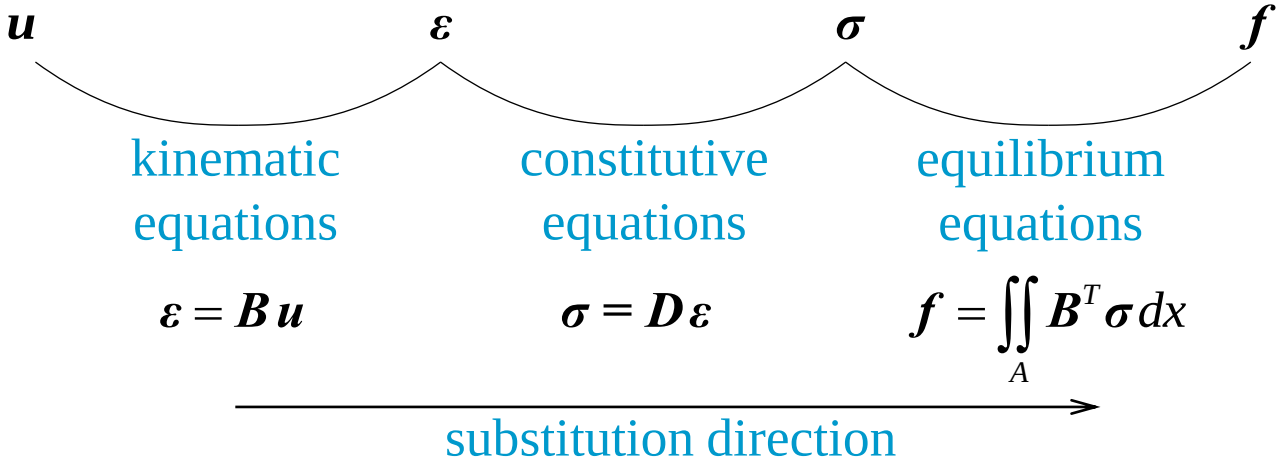
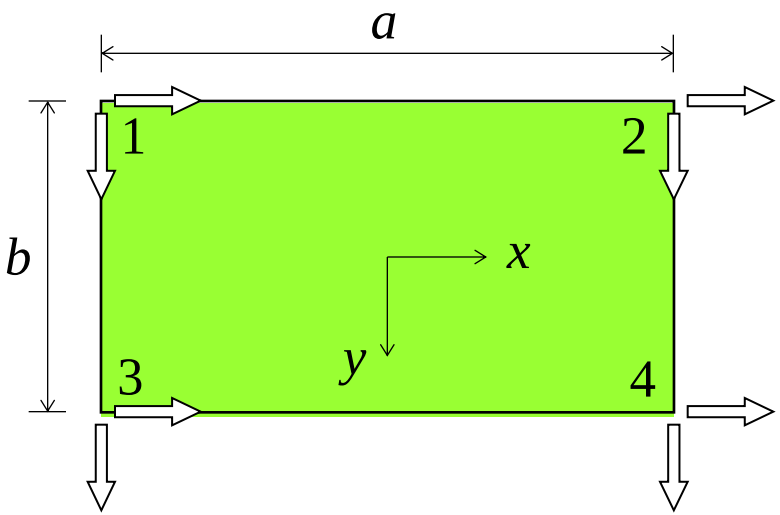
$$\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} = \frac{E\,t}{1-\upsilon^2} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$$

(*constitutive relation*)

$$\boldsymbol{\sigma = D\,\varepsilon = D\,B\,u}$$

$$\begin{Bmatrix} n_{xx} \\ n_{yy} \\ n_{xy} \end{Bmatrix} = \frac{E\,t}{1-\upsilon^2} \left[\begin{array}{cccccccc} -\frac{1}{a}g^+ & -\frac{\upsilon}{b}f^+ & \frac{1}{a}g^+ & -\frac{\upsilon}{b}f^- & -\frac{1}{a}g^- & \frac{\upsilon}{b}f^+ & \frac{1}{a}g^- & \frac{\upsilon}{b}f^- \\ -\frac{\upsilon}{a}g^+ & -\frac{1}{b}f^+ & \frac{\upsilon}{a}g^+ & -\frac{1}{b}f^- & -\frac{\upsilon}{a}g^- & \frac{1}{b}f^+ & \frac{\upsilon}{a}g^- & \frac{1}{b}f^- \\ -\frac{1-\upsilon}{2b}f^+ & -\frac{1-\upsilon}{2a}g^+ & -\frac{1-\upsilon}{2b}f^- & \frac{1-\upsilon}{2a}g^+ & \frac{1-\upsilon}{2b}f^+ & -\frac{1-\upsilon}{2a}g^- & \frac{1-\upsilon}{2b}f^- & \frac{1-\upsilon}{2a}g^- \end{array}\right] \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

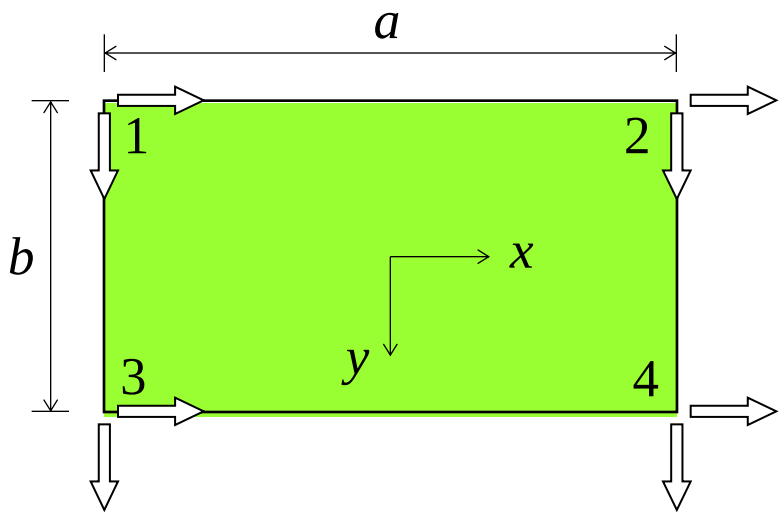




$$\delta \boldsymbol{u}^T \boldsymbol{f} = \iint_A \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dA$$

$$\delta \boldsymbol{\varepsilon} = \boldsymbol{B}(x,y)\, \delta \boldsymbol{u}$$

$$\boldsymbol{f} = \iint_A \boldsymbol{B}^T \boldsymbol{\sigma}\, dA$$



$$\boldsymbol{f} = \iint_A \boldsymbol{B}^T \boldsymbol{\sigma}\, dA$$

$$\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{\varepsilon} = \boldsymbol{D}\,\boldsymbol{B}\,\boldsymbol{u}$$

$$\boldsymbol{f} = \boldsymbol{K}\,\boldsymbol{u}$$

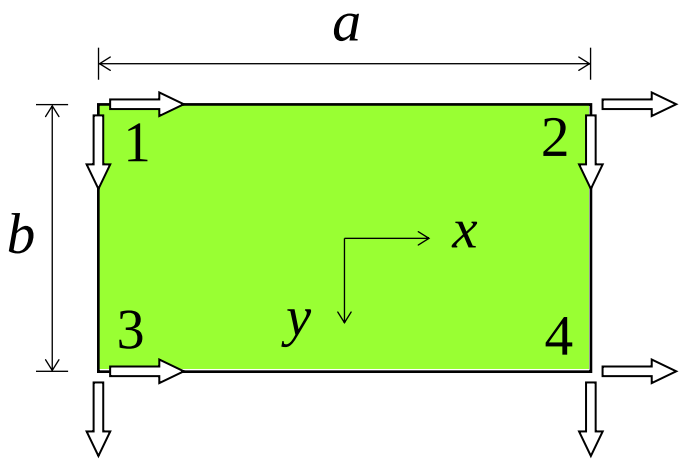
$$\boldsymbol{K} = \iint_A \boldsymbol{B}^T(x,y)\,\boldsymbol{D}(x,y)\,\boldsymbol{B}(x,y)\,dA$$

$$\boldsymbol{K} = \iint_A \boldsymbol{B}^T(x,y)\,\boldsymbol{D}(x,y)\,\boldsymbol{B}(x,y)\,dA$$

$$\begin{bmatrix}
\frac{(g^+)^2}{a^2}+\frac{1-\upsilon}{2}\frac{(f^+)^2}{b^2} & \frac{\upsilon f^+g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^+g^+}{ab} & -\frac{(g^+)^2}{a^2}+\frac{1-\upsilon}{2}\frac{f^-f^+}{b^2} & \frac{\upsilon f^-g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^+g^+}{ab} & \frac{g^-g^+}{a^2}+\frac{1-\upsilon}{2}\frac{(f^+)^2}{b^2} & -\frac{\upsilon f^+g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^+g^-}{ab} & -\frac{g^-g^+}{a^2}+\frac{1-\upsilon}{2}\frac{f^-f^+}{b^2} & -\frac{\upsilon f^-g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^+g^-}{ab} \\
 & \frac{(f^+)^2}{b^2}+\frac{1-\upsilon}{2}\frac{(g^+)^2}{a^2} & -\frac{\upsilon f^+g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^-g^+}{ab} & \frac{f^-f^+}{b^2}+\frac{1-\upsilon}{2}\frac{(g^+)^2}{a^2} & \frac{\upsilon f^+g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^+g^+}{ab} & -\frac{(f^+)^2}{b^2}+\frac{1-\upsilon}{2}\frac{g^-g^+}{a^2} & -\frac{\upsilon f^+g^-}{ab}+\frac{1-\upsilon}{2}\frac{f^-g^+}{ab} & -\frac{f^-f^+}{b^2}+\frac{1-\upsilon}{2}\frac{g^-g^+}{a^2} \\
 & & \frac{(g^+)^2}{a^2}+\frac{1-\upsilon}{2}\frac{(f^-)^2}{b^2} & -\frac{\upsilon f^-g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^-g^+}{ab} & -\frac{g^-g^+}{a^2}+\frac{1-\upsilon}{2}\frac{f^-f^+}{b^2} & \frac{\upsilon f^+g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^-g^-}{ab} & \frac{g^-g^+}{a^2}+\frac{1-\upsilon}{2}\frac{(f^-)^2}{b^2} & \frac{\upsilon f^-g^+}{ab}+\frac{1-\upsilon}{2}\frac{f^-g^-}{ab} \\
 & & & \frac{(f^-)^2}{b^2}+\frac{1-\upsilon}{2}\frac{(g^+)^2}{a^2} & \frac{\upsilon f^-g^-}{ab}+\frac{1-\upsilon}{2}\frac{f^+g^+}{ab} & -\frac{f^-f^+}{b^2}+\frac{1-\upsilon}{2}\frac{g^-g^+}{a^2} & -\frac{\upsilon f^-g^-}{ab}+\frac{1-\upsilon}{2}\frac{f^-g^+}{ab} & -\frac{(f^-)^2}{b^2}+\frac{1-\upsilon}{2}\frac{g^-g^+}{a^2} \\
 & & & & \frac{(g^-)^2}{a^2}+\frac{1-\upsilon}{2}\frac{(f^+)^2}{b^2} & -\frac{\upsilon f^+g^-}{ab}+\frac{1-\upsilon}{2}\frac{f^+g^-}{ab} & -\frac{(g^-)^2}{a^2}+\frac{1-\upsilon}{2}\frac{f^-f^+}{b^2} & \frac{\upsilon f^-g^-}{ab}+\frac{1-\upsilon}{2}\frac{f^+g^-}{ab} \\
 & & & & & \frac{(f^+)^2}{b^2}+\frac{1-\upsilon}{2}\frac{(g^-)^2}{a^2} & \frac{\upsilon f^+g^-}{ab}+\frac{1-\upsilon}{2}\frac{f^-g^-}{ab} & \frac{f^-f^+}{b^2}+\frac{1-\upsilon}{2}\frac{(g^-)^2}{a^2} \\
 & & & & & & \frac{(g^-)^2}{a^2}+\frac{1-\upsilon}{2}\frac{(f^-)^2}{b^2} & \frac{\upsilon f^-g^-}{ab}+\frac{1-\upsilon}{2}\frac{f^-g^-}{ab} \\
 & & & & & & & \frac{(f^-)^2}{b^2}+\frac{1-\upsilon}{2}\frac{(g^-)^2}{a^2}
\end{bmatrix}$$

symmetric

Each term must be multiplied by: $\frac{Et}{1-\upsilon^2}$



$$f^-(x) = \frac{1}{2} + \frac{x}{a} \quad ; \quad g^-(y) = \frac{1}{2} + \frac{y}{b}$$

$$f^+(x) = \frac{1}{2} - \frac{x}{a} \quad ; \quad g^+(y) = \frac{1}{2} - \frac{y}{b}$$

$$\iint_A \left(f^+\right)^2 dA = \iint_A \left(f^-\right)^2 dA = \iint_A \left(g^+\right)^2 dA = \iint_A \left(g^-\right)^2 dA = \frac{1}{3}ab$$

$$\iint_A f^- f^+ dA = \iint_A g^- g^+ dA = \frac{1}{6}ab$$

$$\iint_A f^- g^+ dA = \iint_A f^- g^- dA = \iint_A f^+ g^- dA = \iint_A f^+ g^+ dA = \frac{1}{4}ab$$

$$
\begin{bmatrix}
\frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{6} & \frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & -\frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{12} & \frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & \frac{\beta}{6}-\frac{(1-\upsilon)\alpha}{6} & -\frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & -\frac{\beta}{6}-\frac{(1-\upsilon)\alpha}{12} & -\frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} \\
 & \frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{6} & -\frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & \frac{\alpha}{6}-\frac{(1-\upsilon)\beta}{6} & \frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & -\frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{12} & -\frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & -\frac{\alpha}{6}-\frac{(1-\upsilon)\beta}{12} \\
 & & \frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{6} & -\frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & -\frac{\beta}{6}-\frac{(1-\upsilon)\alpha}{12} & \frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & \frac{\beta}{6}-\frac{(1-\upsilon)\alpha}{6} & \frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} \\
 & & & \frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{6} & \frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & -\frac{\alpha}{6}-\frac{(1-\upsilon)\beta}{12} & -\frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} & -\frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{12} \\
 & \text{symmetric} & & & \frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{6} & -\frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & -\frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{12} & -\frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} \\
 & & & & & \frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{6} & \frac{\upsilon}{4}-\frac{(1-\upsilon)}{8} & \frac{\alpha}{6}-\frac{(1-\upsilon)\beta}{6} \\
 & & & & & & \frac{\beta}{3}+\frac{(1-\upsilon)\alpha}{6} & \frac{\upsilon}{4}+\frac{(1-\upsilon)}{8} \\
 & & & & & & & \frac{\alpha}{3}+\frac{(1-\upsilon)\beta}{6}
\end{bmatrix}
$$

Each term must be multiplied by: $\frac{Et}{1-\upsilon^2}$

$$\alpha = \beta^{-1} = \frac{a}{b}$$

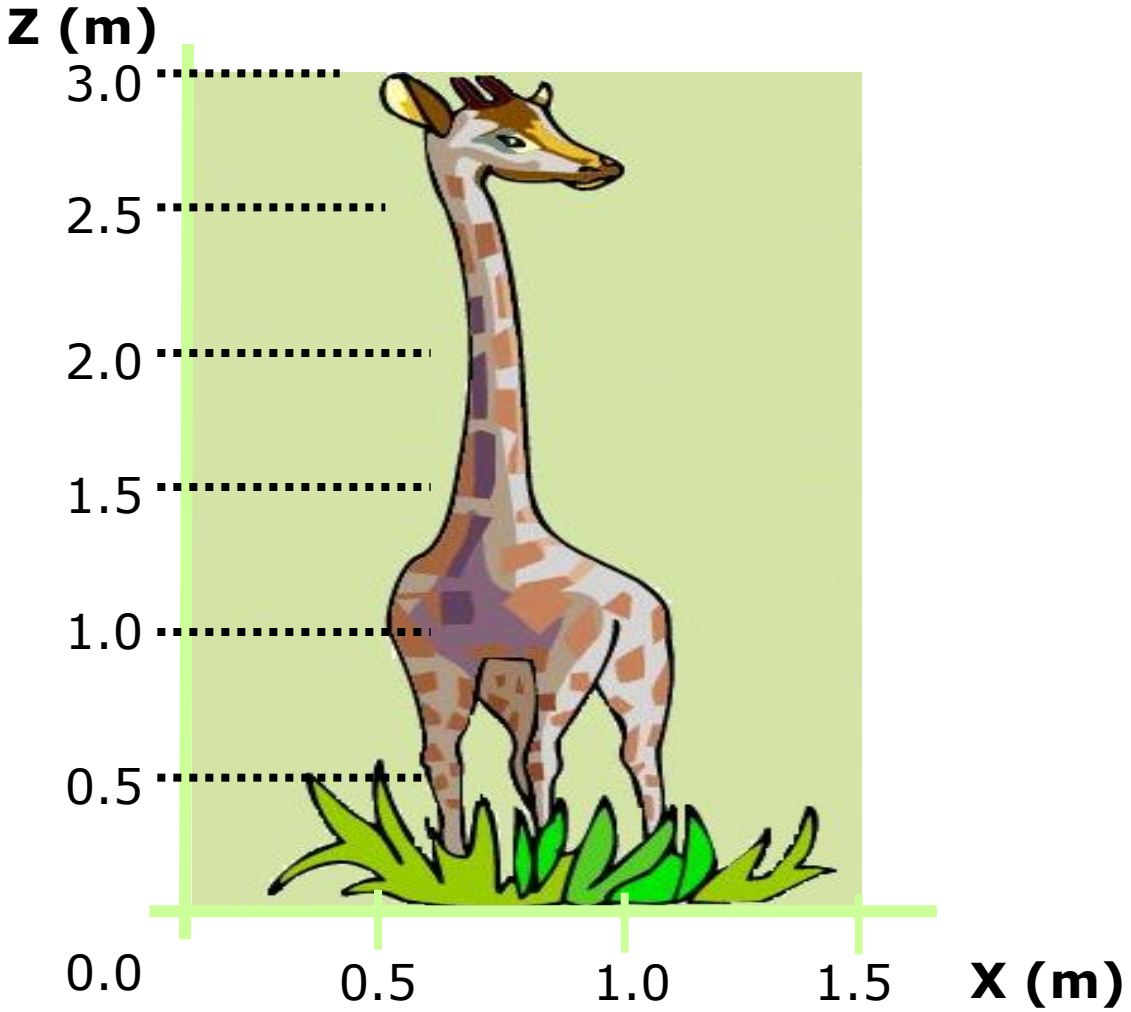
$$\boldsymbol{K} = \iint_A \boldsymbol{B}^T(x,y)\,\boldsymbol{D}(x,y)\,\boldsymbol{B}(x,y)\,dA$$

TU Delft

$$\boldsymbol{\sigma = D\varepsilon = DBu}$$

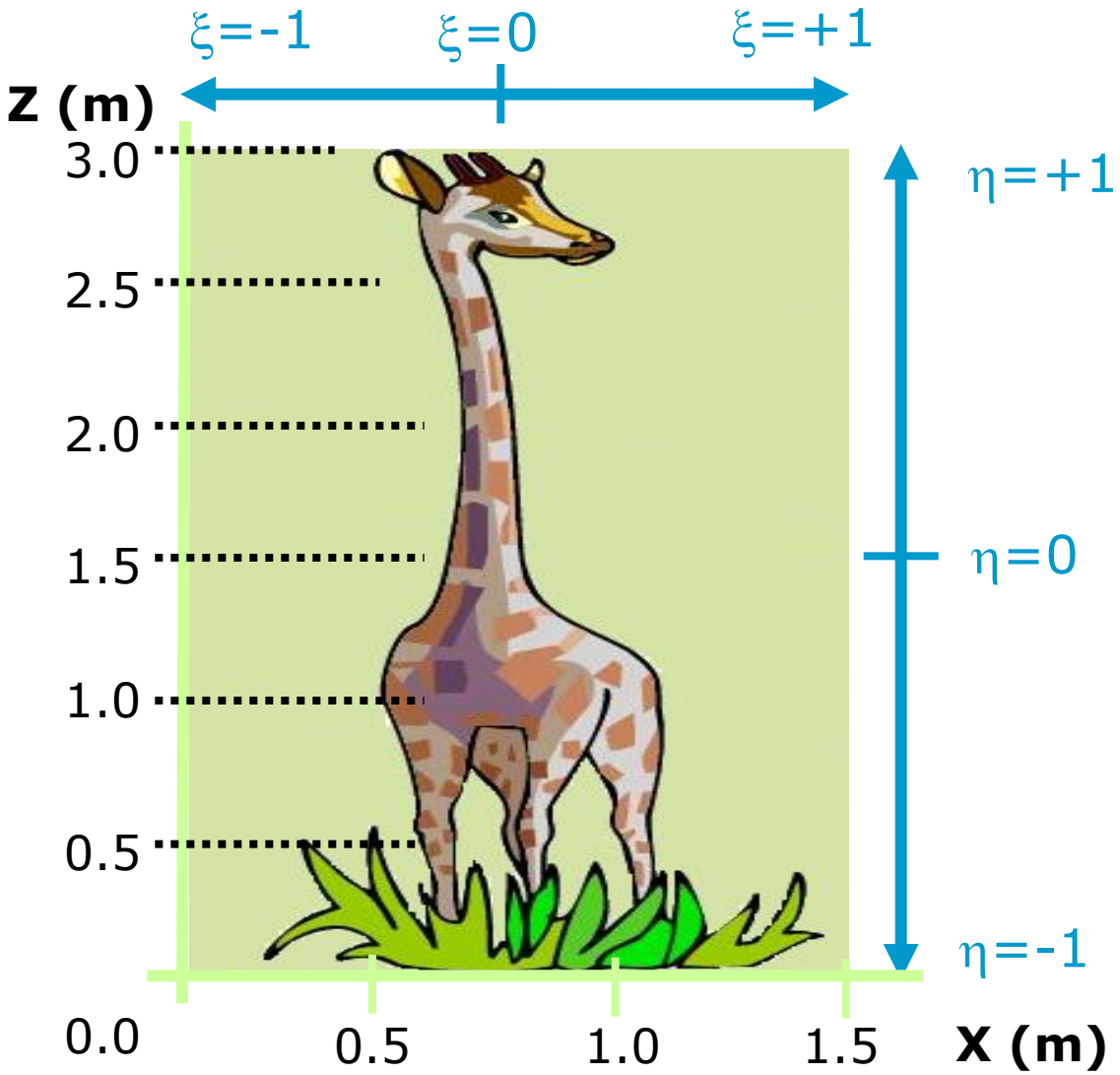
$$\begin{Bmatrix} n_{xx_1} \\ n_{yy_1} \\ n_{xy_1} \\ \hline n_{xx_2} \\ n_{yy_2} \\ n_{xy_2} \\ \hline n_{xx_3} \\ n_{yy_3} \\ n_{xy_3} \\ \hline n_{xx_4} \\ n_{yy_4} \\ n_{xy_4} \end{Bmatrix} = \frac{Et}{1-\upsilon^2} \left[\begin{array}{cccccccc} -\frac{1}{a} & -\frac{\upsilon}{b} & \frac{1}{a} & 0 & 0 & \frac{\upsilon}{b} & 0 & 0 \\ -\frac{\upsilon}{a} & -\frac{1}{b} & \frac{\upsilon}{a} & 0 & 0 & \frac{1}{b} & 0 & 0 \\ -\frac{1-\upsilon}{2b} & -\frac{1-\upsilon}{2a} & 0 & \frac{1-\upsilon}{2a} & \frac{1-\upsilon}{2b} & 0 & 0 & 0 \\ \hline -\frac{1}{a} & 0 & \frac{1}{a} & -\frac{\upsilon}{b} & 0 & 0 & 0 & \frac{\upsilon}{b} \\ -\frac{\upsilon}{a} & 0 & \frac{\upsilon}{a} & -\frac{1}{b} & 0 & 0 & 0 & \frac{1}{b} \\ 0 & -\frac{1-\upsilon}{2a} & -\frac{1-\upsilon}{2b} & \frac{1-\upsilon}{2a} & 0 & 0 & \frac{1-\upsilon}{2b} & 0 \\ \hline 0 & -\frac{\upsilon}{b} & 0 & 0 & -\frac{1}{a} & \frac{\upsilon}{b} & \frac{1}{a} & 0 \\ 0 & -\frac{1}{b} & 0 & 0 & -\frac{\upsilon}{a} & \frac{1}{b} & \frac{\upsilon}{a} & 0 \\ -\frac{1-\upsilon}{2b} & 0 & 0 & 0 & \frac{1-\upsilon}{2b} & -\frac{1-\upsilon}{2a} & 0 & \frac{1-\upsilon}{2a} \\ \hline 0 & 0 & 0 & -\frac{\upsilon}{b} & -\frac{1}{a} & 0 & \frac{1}{a} & \frac{\upsilon}{b} \\ 0 & 0 & 0 & -\frac{1}{b} & -\frac{\upsilon}{a} & 0 & \frac{\upsilon}{a} & \frac{1}{b} \\ 0 & 0 & -\frac{1-\upsilon}{2b} & 0 & 0 & -\frac{1-\upsilon}{2a} & \frac{1-\upsilon}{2b} & \frac{1-\upsilon}{2a} \end{array}\right] \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{Bmatrix}$$

Coordinate transformation

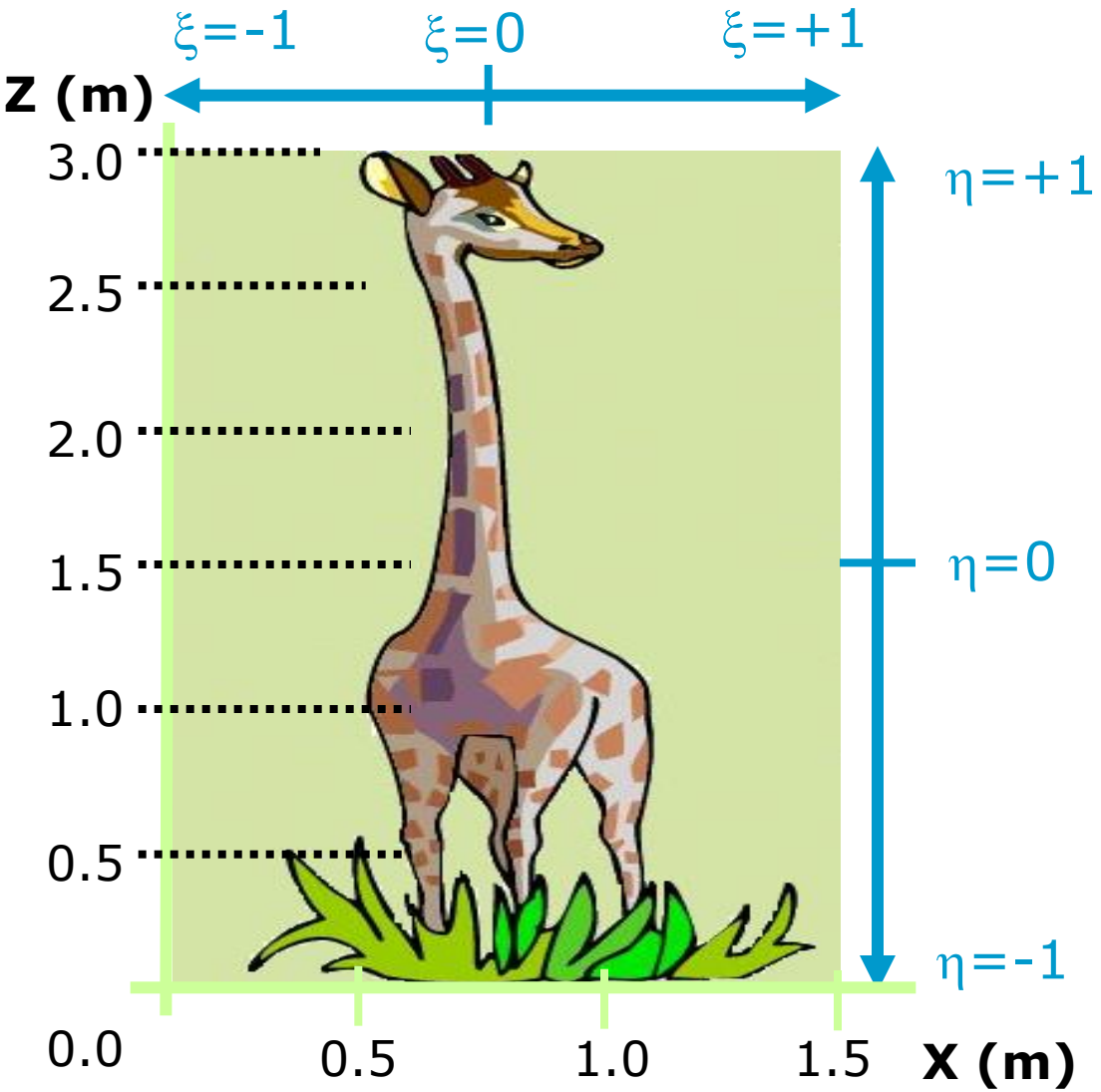


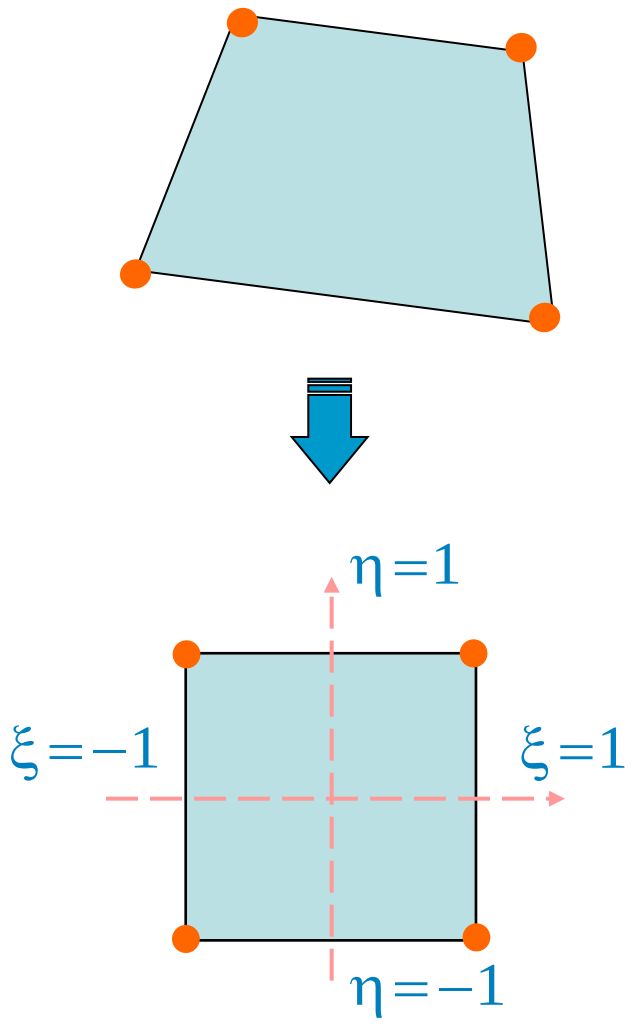
Giraffe

Coordinate transformation

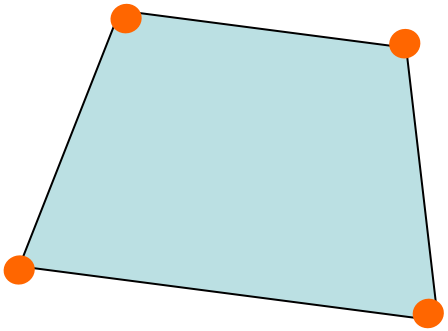


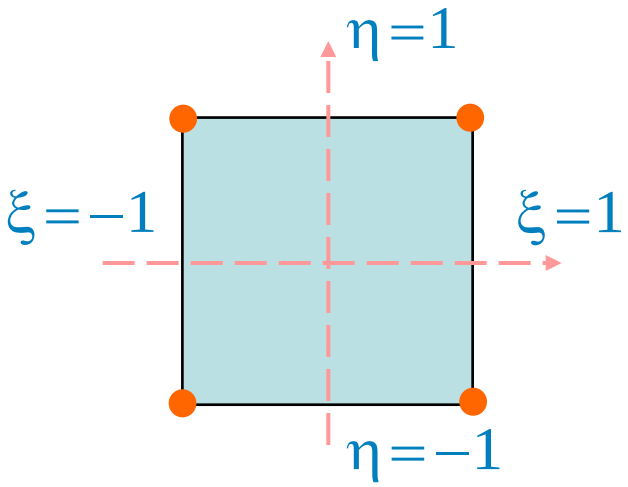
Coordinate transformation

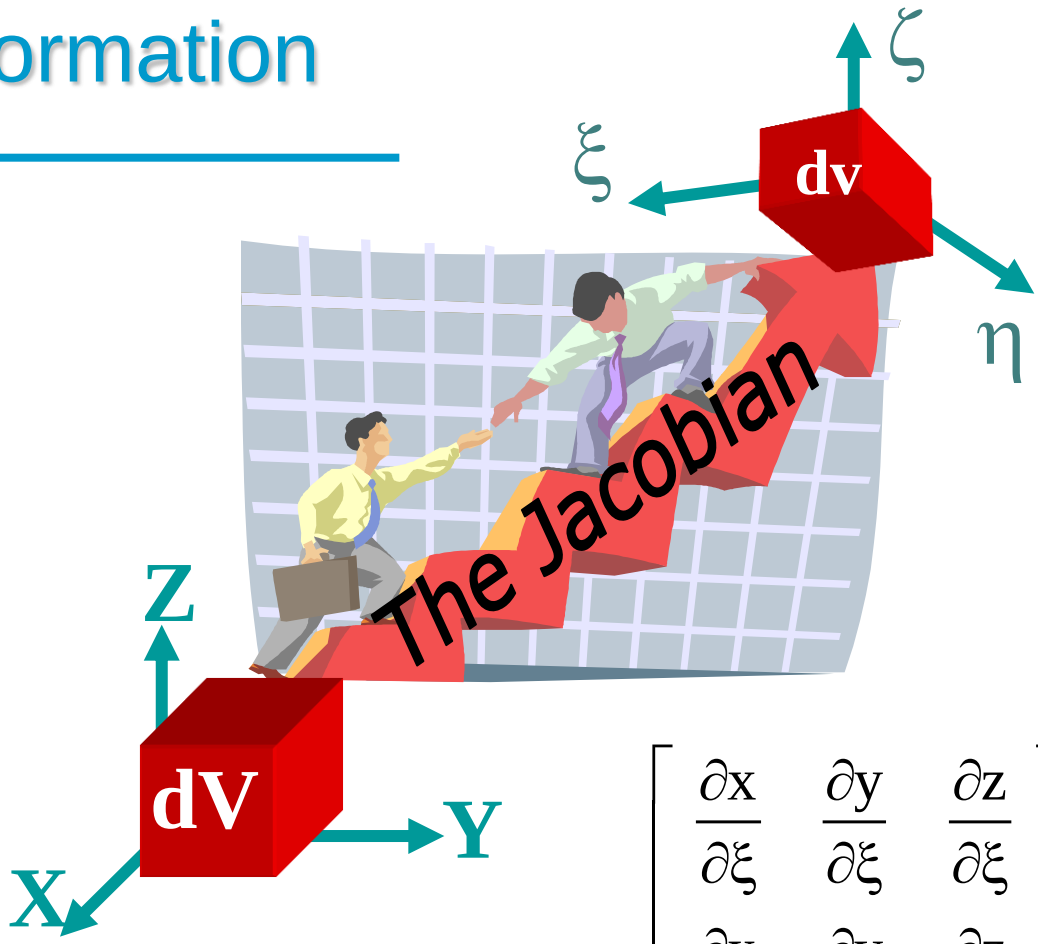




Coordinate transformation



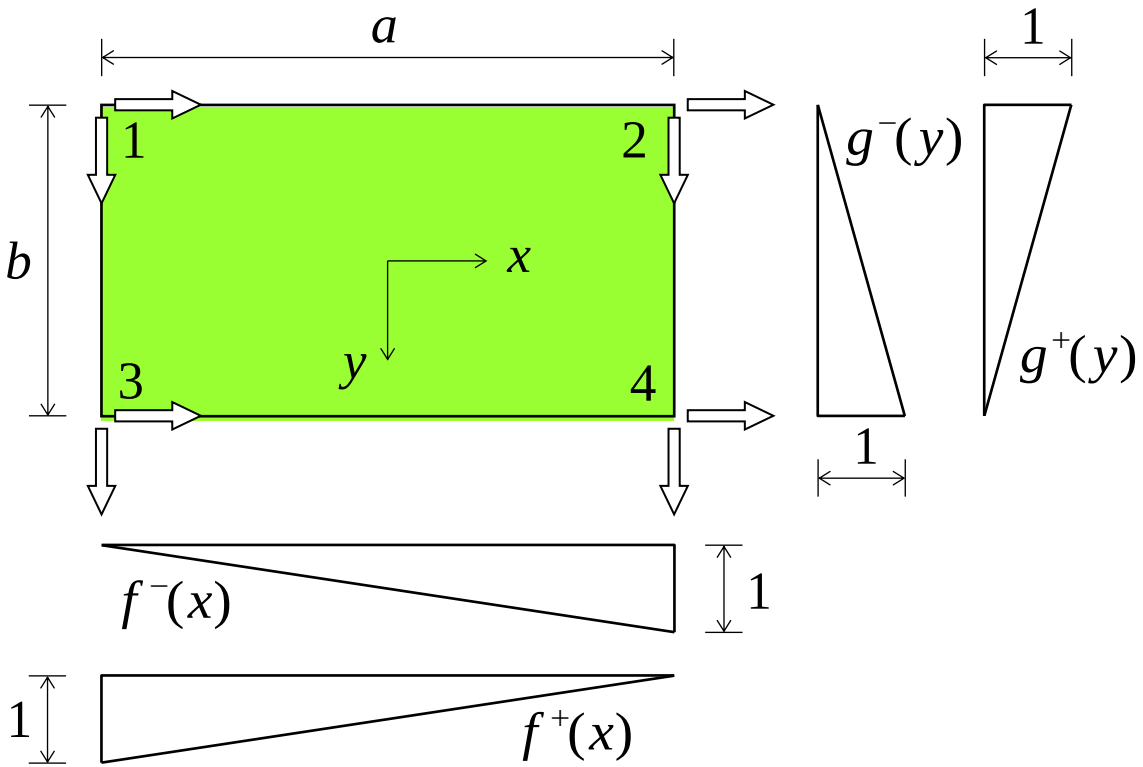


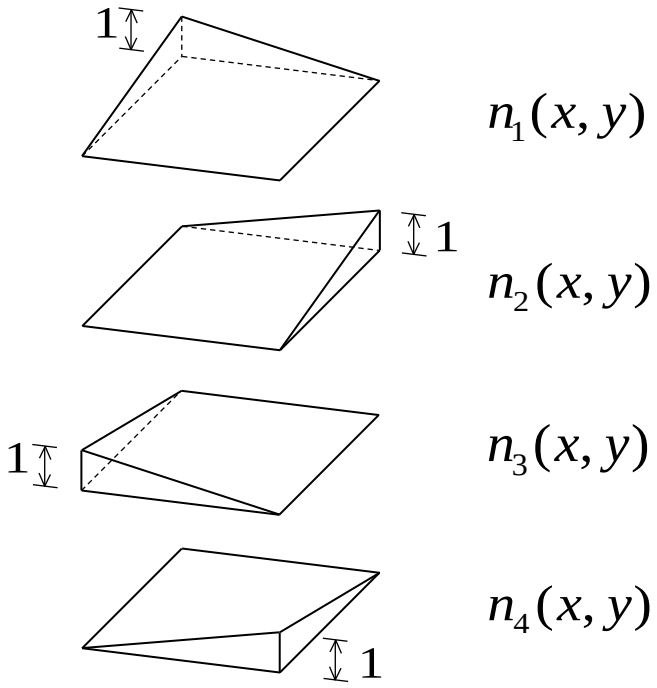


$$dV = \det J \, dv$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

TU Delft





$$f^-(x) = \frac{1}{2} + \frac{x}{a} \quad ; \quad g^-(y) = \frac{1}{2} + \frac{y}{b}$$

$$f^+(x) = \frac{1}{2} - \frac{x}{a} \quad ; \quad g^+(y) = \frac{1}{2} - \frac{y}{b}$$

$$n_1(x,y) = f^+(x) \cdot g^+(y)$$

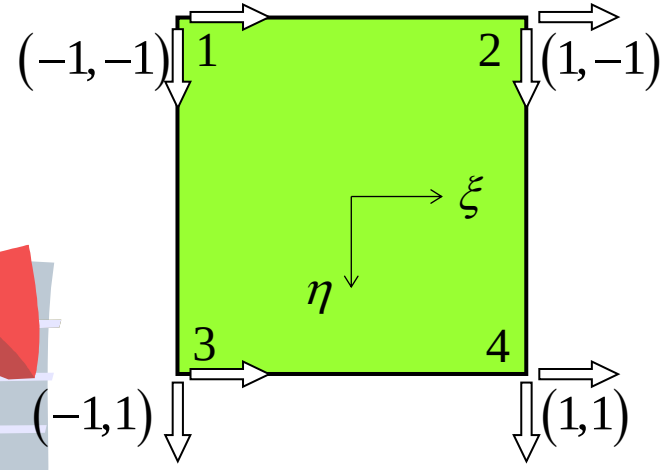
$$n_2(x,y) = f^-(x) \cdot g^+(y)$$

$$n_3(x,y) = f^+(x) \cdot g^-(y)$$

$$n_4(x,y) = f^-(x) \cdot g^-(y)$$

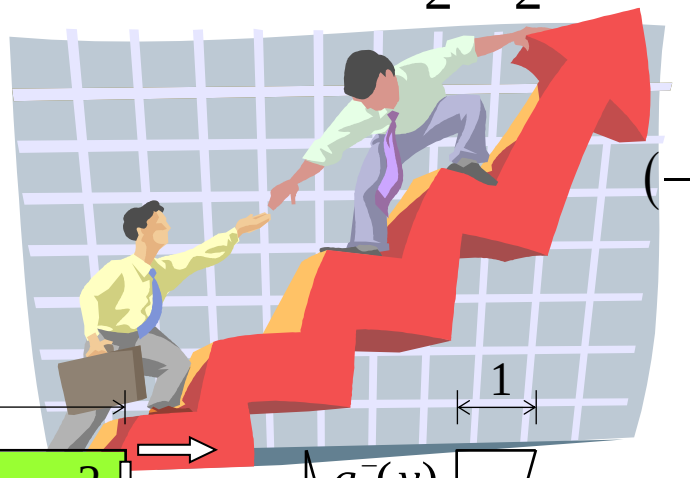
$$f^-(\xi) = \frac{1}{2} + \frac{\xi}{2}$$

$$f^+(\xi) = \frac{1}{2} - \frac{\xi}{2}$$



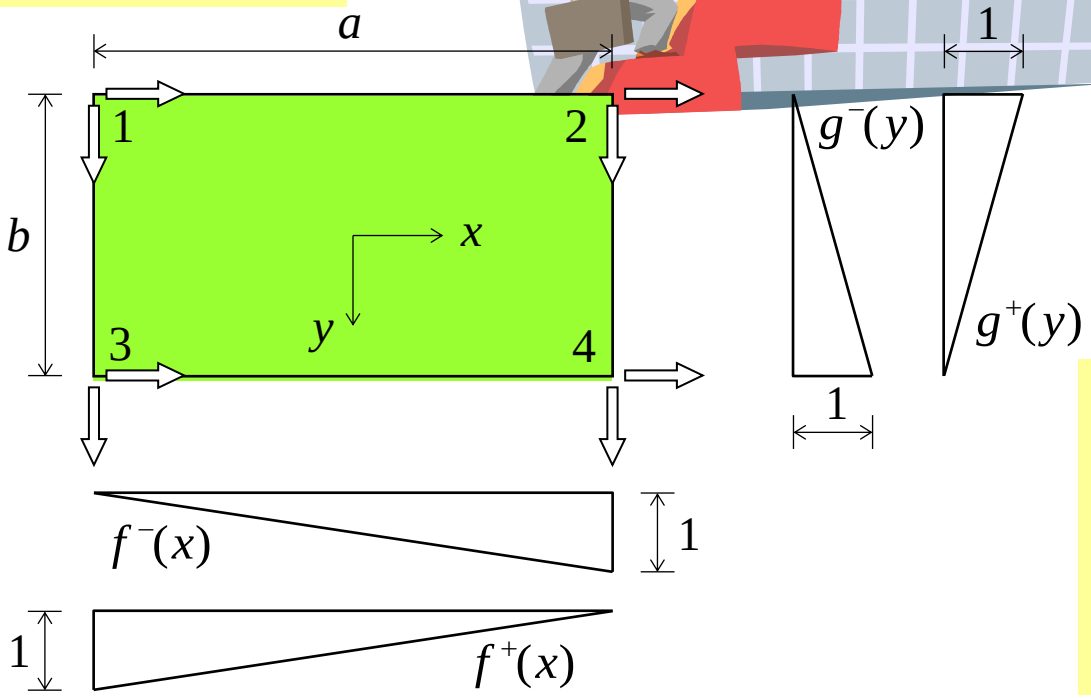
$$f^-(x) = \frac{1}{2} + \frac{x}{a}$$

$$f^+(x) = \frac{1}{2} - \frac{x}{a}$$



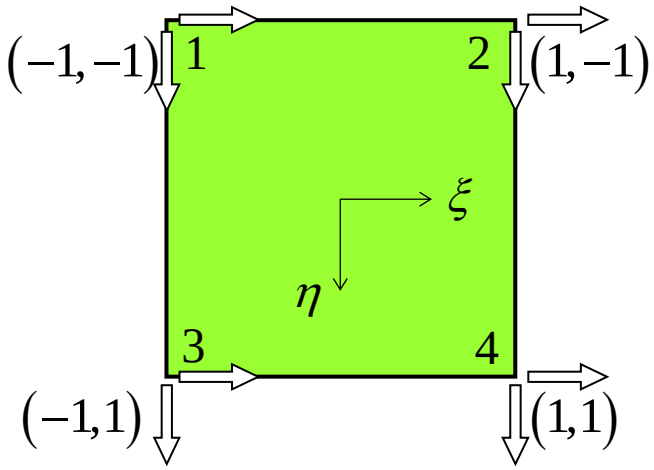
$$g^-(\eta) = \frac{1}{2} + \frac{\eta}{2}$$

$$g^+(\eta) = \frac{1}{2} - \frac{\eta}{2}$$



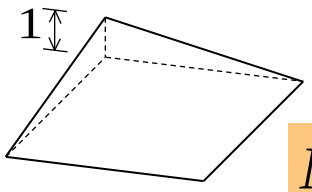
$$g^-(y) = \frac{1}{2} + \frac{y}{b}$$

$$g^+(y) = \frac{1}{2} - \frac{y}{b}$$

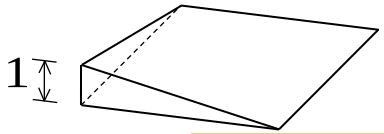


$$g^-(\eta) = \frac{1}{2} + \frac{\eta}{2} \qquad g^+(\eta) = \frac{1}{2} - \frac{\eta}{2}$$

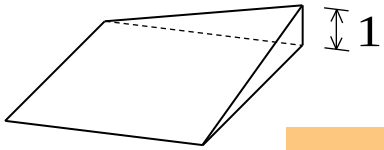
$$f^-(\xi) = \frac{1}{2} + \frac{\xi}{2} \qquad f^+(\xi) = \frac{1}{2} - \frac{\xi}{2}$$



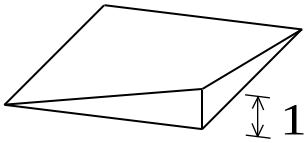
$$N_1(\xi,\eta) = f^+(\xi)\cdot g^+(\eta)$$



$$N_3(\xi,\eta) = f^+(\xi)\cdot g^-(\eta)$$



$$N_2(\xi,\eta) = f^-(\xi)\cdot g^+(\eta)$$



$$N_4(\xi,\eta) = f^-(\xi)\cdot g^-(\eta)$$

TU Delft

- coordinate interpolation

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \left[\begin{array}{cc|cc|cc} N_1 & 0 & N_2 & 0 & \cdots & \cdots \\ 0 & N_1 & 0 & N_2 & \cdots & \cdots \end{array}\right] \left\{\begin{array}{c} a_{x1} \\ a_{y1} \\ \hline a_{x2} \\ a_{y2} \\ \hline \vdots \\ \vdots \end{array}\right\}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{bmatrix} a_{xi} \\ a_{yi} \end{bmatrix} = \sum_{i=1}^{n} \mathbf{N}_i \, \mathbf{a}_i = \mathbf{N}\,\mathbf{a}$$

- displacement interpolation

$$\mathbf{u} = \begin{Bmatrix} u_x \\ u_v \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{bmatrix} d_{xi} \\ d_{yi} \end{bmatrix} = \sum_{i=1}^{n} \mathbf{N}_i \, \mathbf{d}_i = \mathbf{N}\,\mathbf{d}$$

TU Delft

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \left[\begin{array}{cc|cc|cc} N_1 & 0 & N_2 & 0 & \dots & \dots \\ 0 & N_1 & 0 & N_2 & \dots\dots & \end{array}\right] \begin{Bmatrix} d_{x1} \\ d_{y1} \\ d_{x2} \\ d_{y2} \\ \vdots \\ \vdots \end{Bmatrix}$$

$$= \mathbf{L} \left[\mathbf{N}_1 \mid \mathbf{N}_2 \mid \dots \dots \right] \mathbf{d}$$

$$\mathbf{B}_i = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & 0 \\ 0 & \dfrac{\partial N_i}{\partial y} \\ \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_i}{\partial x} \end{bmatrix}$$

$$\left[\mathbf{B}_1 \quad \mathbf{B}_2 \quad \dots \quad \dots \right] \mathbf{d}$$

TU Delft

$$\mathbf{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial y}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

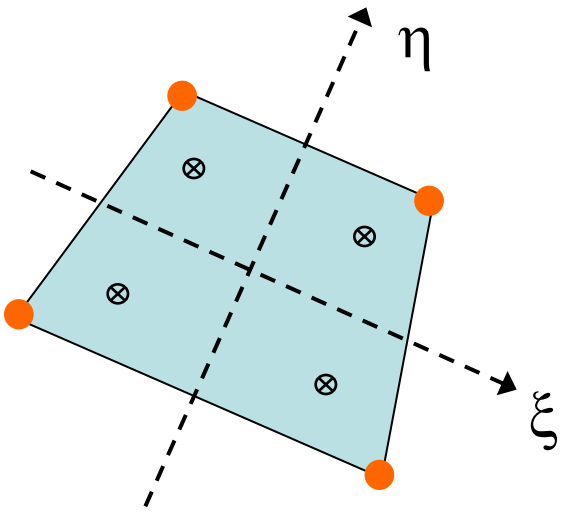
$$[\mathbf{J}]^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \sum_{i=1}^{n} \mathbf{N}_i \, \mathbf{a}_i = \mathbf{N}\,\mathbf{a}$$

Numerical integration

$$\mathbf{k}_{ij} = \int_{\Omega} \left[\mathbf{B}_i \right]^T \mathbf{D} \mathbf{B}_j \, d\Omega = \int_{-1}^{1} \int_{-1}^{1} \left[\mathbf{B}_i \right]^T \mathbf{D} \mathbf{B}_j \, h \det \mathbf{J} \, d\xi \, d\eta$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{m} \Upsilon(\xi_k, \eta_l) W_k W_l$$



$$\mathbf{f}_{b_i} = \int_{\Omega} \left[\mathbf{N}_i \right]^T \mathbf{b} \, d\Omega$$

$$= \int_{-1}^{1} \int_{-1}^{1} \left[\mathbf{N}_i \right]^T \mathbf{b} \, h \det \mathbf{J} \, d\xi \, d\eta$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{m} \Phi(\xi_k, \eta_l) W_k W_l$$

TU Delft