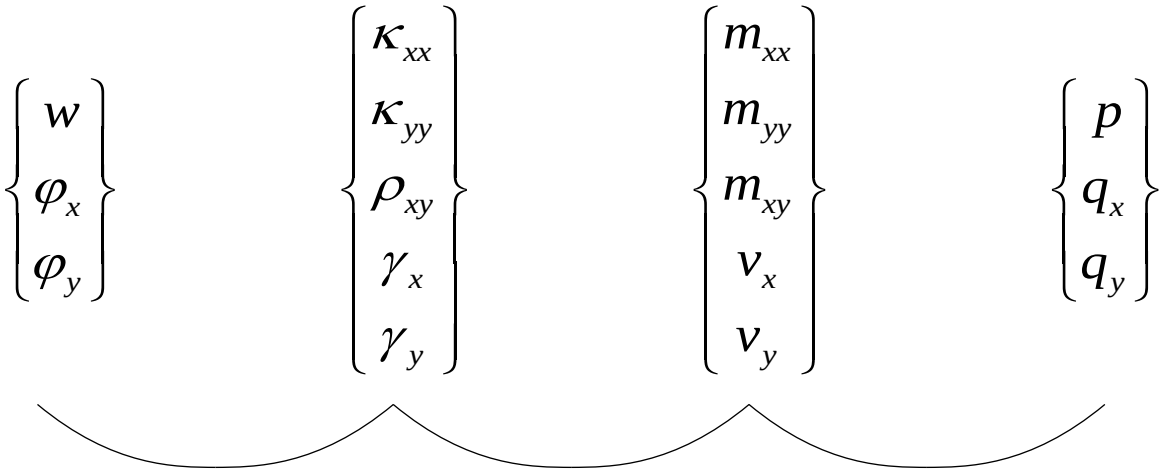
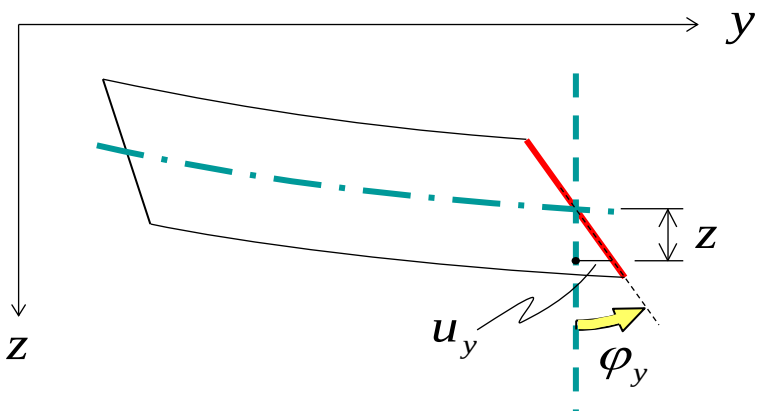


$$\begin{bmatrix} -D_\gamma\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) & -D_\gamma\frac{\partial}{\partial x} & -D_\gamma\frac{\partial}{\partial y} \\ D_\gamma\frac{\partial}{\partial x} & D_\gamma - D\frac{\partial^2}{\partial x^2} - \frac{1}{2}(1-\upsilon)D\frac{\partial^2}{\partial y^2} & -\frac{1}{2}(1+\upsilon)D\frac{\partial^2}{\partial x\partial y} \\ D_\gamma\frac{\partial}{\partial y} & -\frac{1}{2}(1+\upsilon)\frac{\partial^2}{\partial x\partial y} & D_\gamma - \frac{1}{2}(1-\upsilon)D\frac{\partial^2}{\partial x^2} - D\frac{\partial^2}{\partial y^2} \end{bmatrix} \begin{Bmatrix} w \\ \varphi_x \\ \varphi_y \end{Bmatrix} = \begin{Bmatrix} p \\ q_x \\ q_y \end{Bmatrix}$$

Suppositions

•No membrane forces occur, so the mid-plane will remain strainless after loading. This is valid if $w << t$;

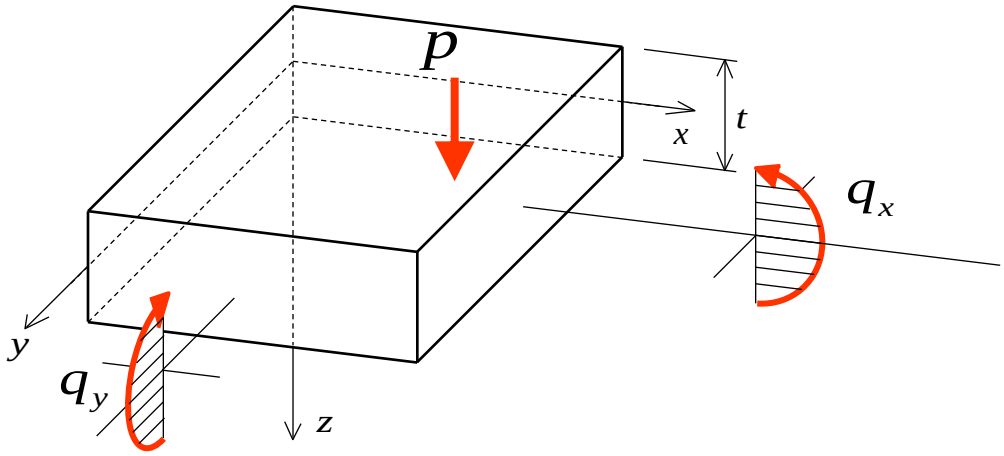
• The "needle hypothesis" is applied on the plane cross-section of the plate;

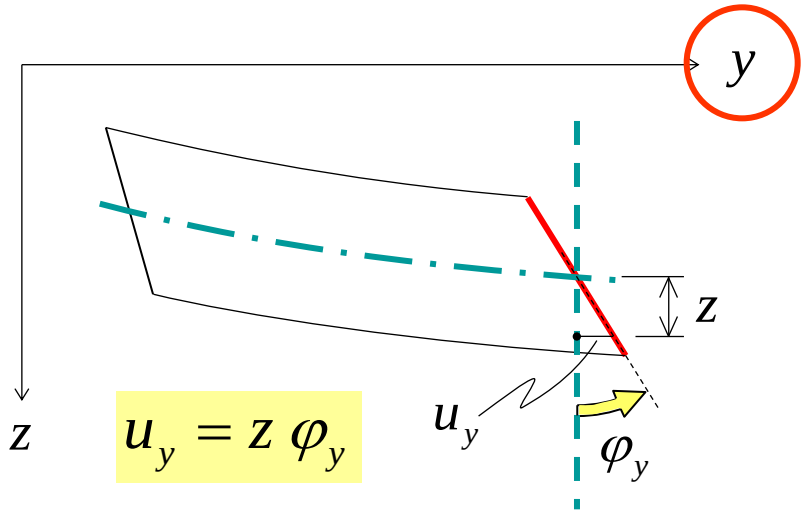


• $\sigma_{zz} = 0$ and ε_{zz} is ignored, no variation of the vertical displacement occur over the thickness of the slab;

• The above three suppositions are valid both for thin and thick plates. For thin plates, the shear deformation is neglected.

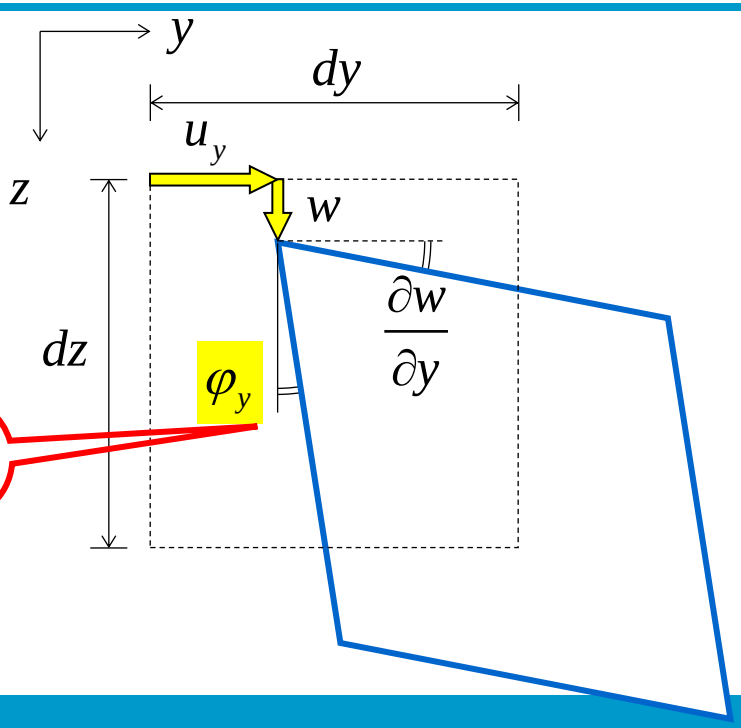
Slender slab kinematic equations





$$\varphi_y = -\frac{\partial w}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \varphi_y$$



Slender slab kinematic equations

φ_x ; φ_y depend on w

$$\varphi_x = -\frac{\partial w}{\partial x} \qquad \varphi_y = -\frac{\partial w}{\partial y}$$

$$\varepsilon_{xx} = z\,\kappa_{xx} = z\frac{\partial \varphi_x}{\partial x}$$

$$\varepsilon_{yy} = z\,\kappa_{yy} = z\frac{\partial \varphi_y}{\partial y}$$

$$\gamma_{xy} = z\,\rho_{xy} = z\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right)$$

$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}$$

$$\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

$$\rho_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} = 2\kappa_{xy}$$

Slender slab

$$\varphi_x = -\frac{\partial w}{\partial x}$$

$$\varphi_y = -\frac{\partial w}{\partial y}$$

$$\begin{Bmatrix} w \\ \cancel{\varphi_x} \\ \cancel{\varphi_y} \end{Bmatrix} \quad \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \\ \cancel{\gamma_x} \\ \cancel{\gamma_y} \end{Bmatrix} \quad \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \\ \cancel{v_x} \\ \cancel{v_y} \end{Bmatrix} \quad \begin{Bmatrix} p \\ \cancel{q_x} \\ \cancel{q_y} \end{Bmatrix}$$

from moment equilibrium

Slab constitutive equations

$$m_{xx} = D\left(\kappa_{xx} + \upsilon\,\kappa_{yy}\right)$$

$$m_{yy} = D\left(\kappa_{yy} + \upsilon\,\kappa_{xx}\right)$$

$$m_{xy} = D\left(1-\upsilon\right)\kappa_{xy}$$

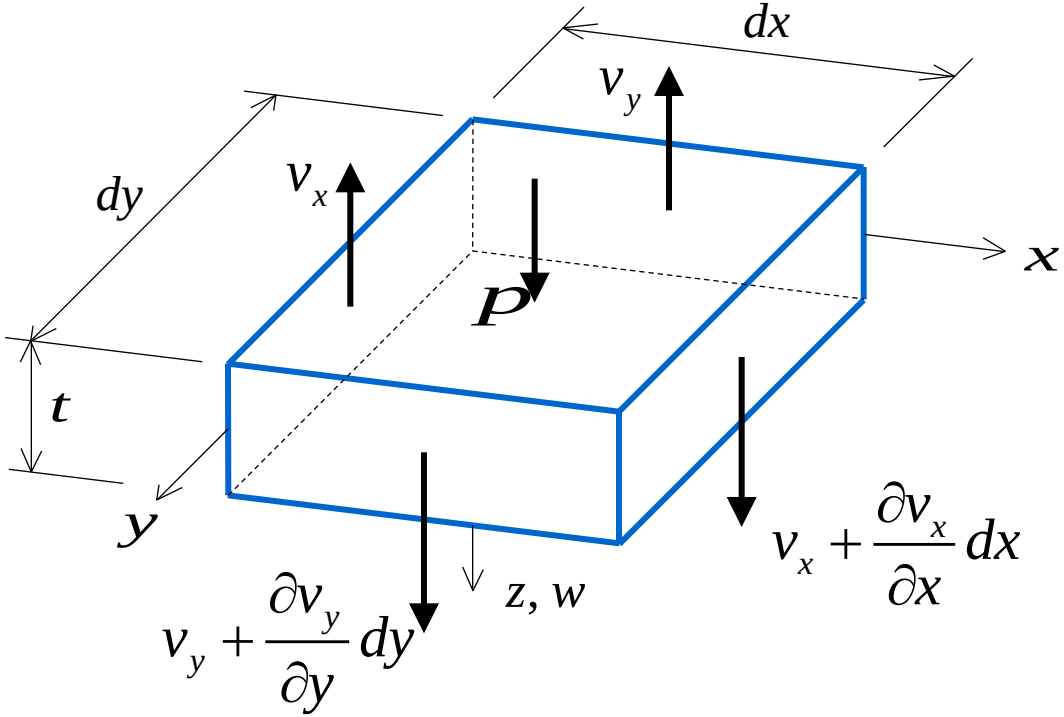
$$\cancel{v_x = D_\gamma\,\gamma_x}$$

$$\cancel{v_y = D_\gamma\,\gamma_y}$$

$$D = \frac{E\,t^3}{12\left(1-\upsilon^2\right)}$$

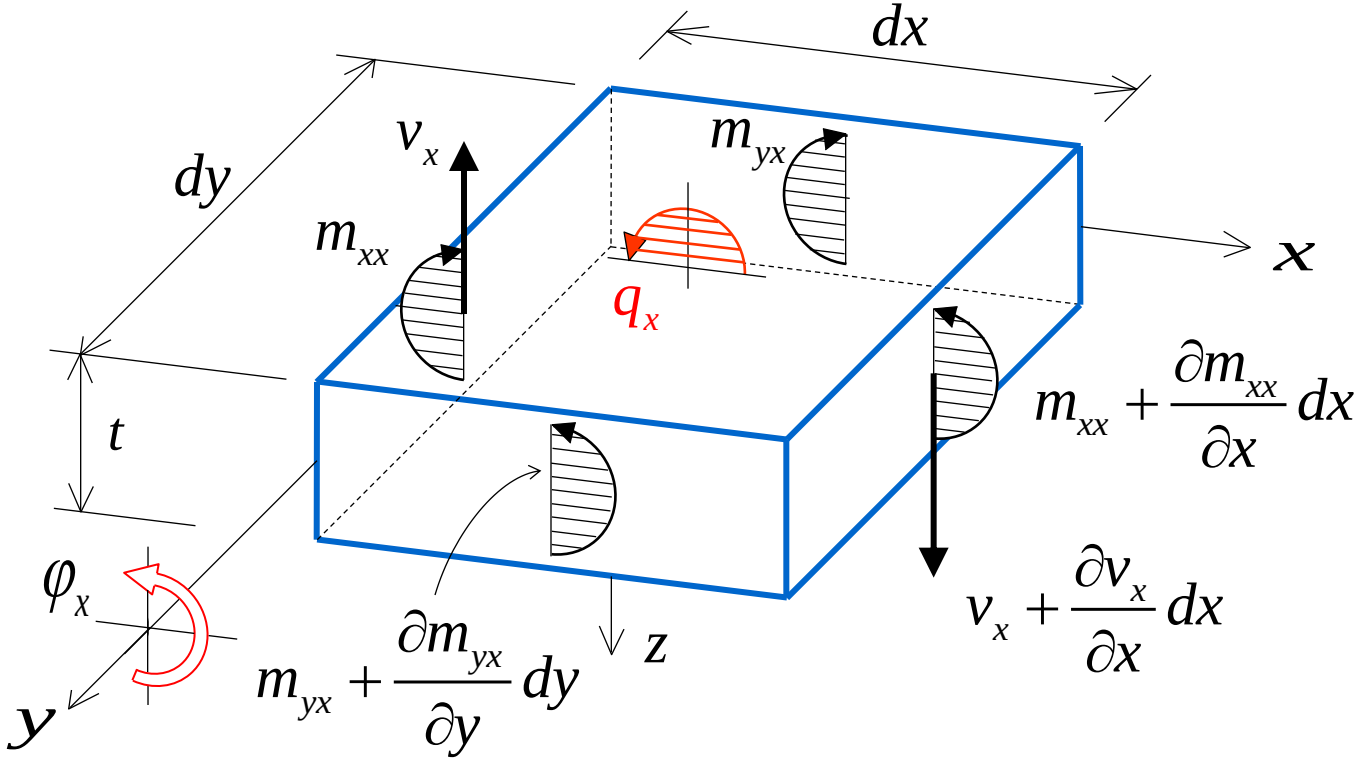
plate stiffness

Slender slab equilibrium equations



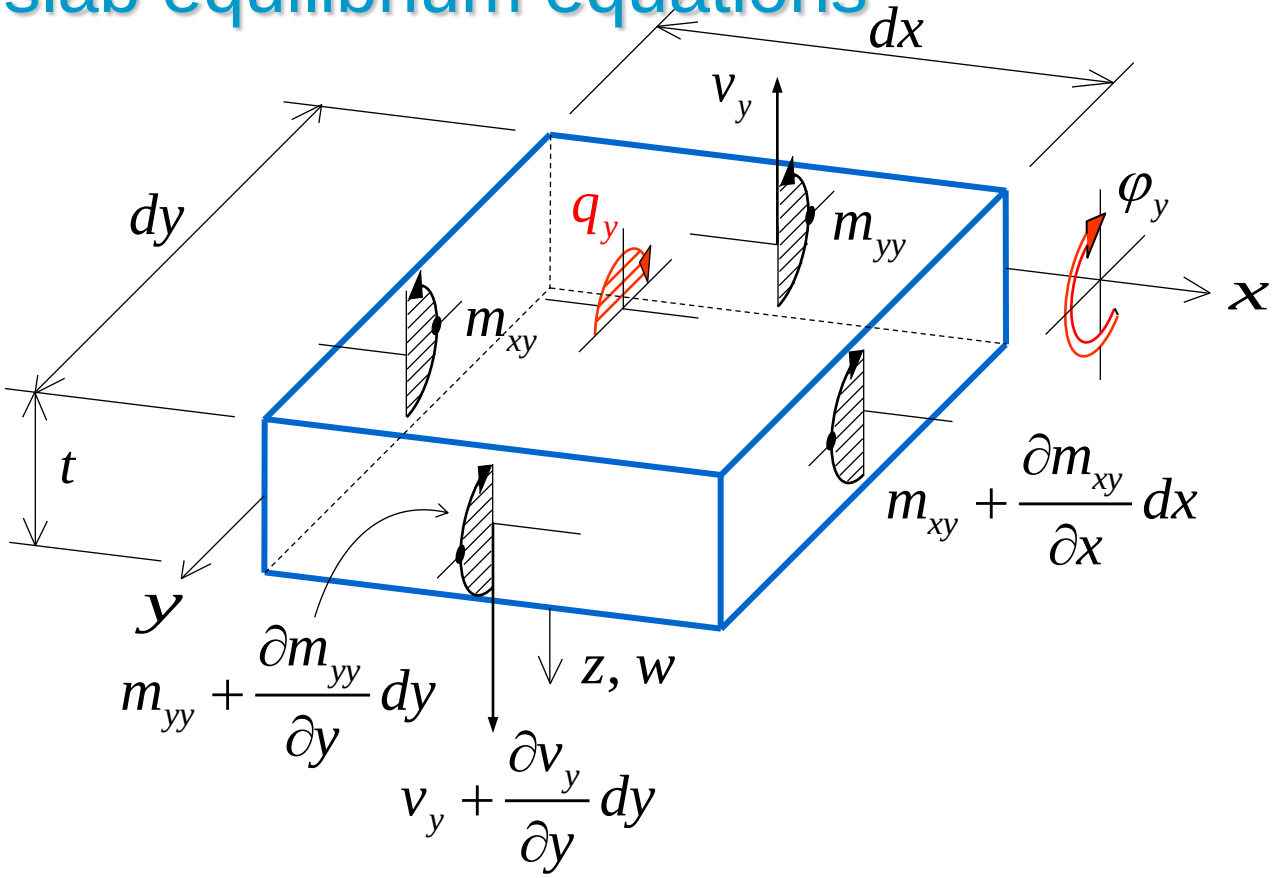
$$\sum f_z = 0 \quad : \quad \left(\frac{\partial v_x}{\partial x} dx \right) dy + \left(\frac{\partial v_y}{\partial y} dy \right) dx + p\, dx\, dy = 0$$

Slender slab equilibrium equations



$$\sum m_y = 0 \quad : \quad \left(\frac{\partial m_{xx}}{\partial x} dx\right) dy + \left(\frac{\partial m_{yx}}{\partial y} dy\right) dx - (v_x dy) dx + q_x dx\, dy = 0$$

Slender slab equilibrium equations



$$\sum m_x = 0 \quad : \quad \left(\frac{\partial m_{yy}}{\partial y} dy\right) dx + \left(\frac{\partial m_{xy}}{\partial x} dx\right) dy - \left(v_y dx\right) dy + q_y dx\, dy = 0$$

Slender slab equilibrium equation(s)

from force equilibrium

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + p = 0$$

$$v_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y}$$

$$v_y = \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

$$-\left(\frac{\partial^2 m_{xx}}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2} \right) = p$$

from moment equilibrium

Slender slab

$$\{w\} \qquad \begin{Bmatrix} \kappa_{xx} \\ \kappa_{yy} \\ \rho_{xy} \end{Bmatrix} \qquad \begin{Bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \end{Bmatrix} \qquad \{p\}$$

kinematic equations | constitutive equations | equilibrium equation

$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}$$

$$\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

$$\rho_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} = 2\kappa_{xy}$$

$$m_{xx} = D\left(\kappa_{xx} + \upsilon\, \kappa_{yy}\right)$$

$$m_{yy} = D\left(\kappa_{yy} + \upsilon\, \kappa_{xx}\right)$$

$$m_{xy} = \frac{1}{2} D\left(1-\upsilon\right) \rho_{xy}$$

$$-\left(\frac{\partial^2 m_{xx}}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2} \right) = p$$

Slender slab

$$m_{xx} = D\left(\kappa_{xx} + \upsilon\,\kappa_{yy}\right)$$

$$m_{yy} = D\left(\kappa_{yy} + \upsilon\,\kappa_{xx}\right)$$

$$m_{xy} = \frac{1}{2} D\left(1-\upsilon\right)\rho_{xy}$$

$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}$$

$$\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

$$\rho_{xy} = -2\frac{\partial^2 w}{\partial x \partial y} = 2\kappa_{xy}$$

$$m_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{yy} = -D\left(\upsilon\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{xy} = -\left(1-\upsilon\right) D\frac{\partial^2 w}{\partial x \partial y}$$

$$-\left(\frac{\partial^2 m_{xx}}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2}\right) = p$$

Slender slab

$$D\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)w=p$$

Laplace operator: $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

$$D\nabla^2\nabla^2 w=p$$

Biharmonic equation

Slender slab

$$m_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{xy} = -(1-\upsilon) D \frac{\partial^2 w}{\partial x \partial y}$$

$$v_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y}$$

$$v_x = -D\left(\frac{\partial^3 w}{\partial x^3} + \upsilon \frac{\partial^3 w}{\partial x \partial y^2}\right) - (1-\upsilon) D \frac{\partial^3 w}{\partial x \partial y^2}$$

$$= -D\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right) = -D \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w$$

$$= -D \frac{\partial}{\partial x} \nabla^2 w$$

Also:

$$v_y = -D \frac{\partial}{\partial y} \nabla^2 w$$



Plate Analysis: Theory & Applications

CIE4180

Slender Slabs

Xueyan Liu S2 2.26 Cor Kasbergen 6.52 Alexander Schmets 6.63

TU Delft