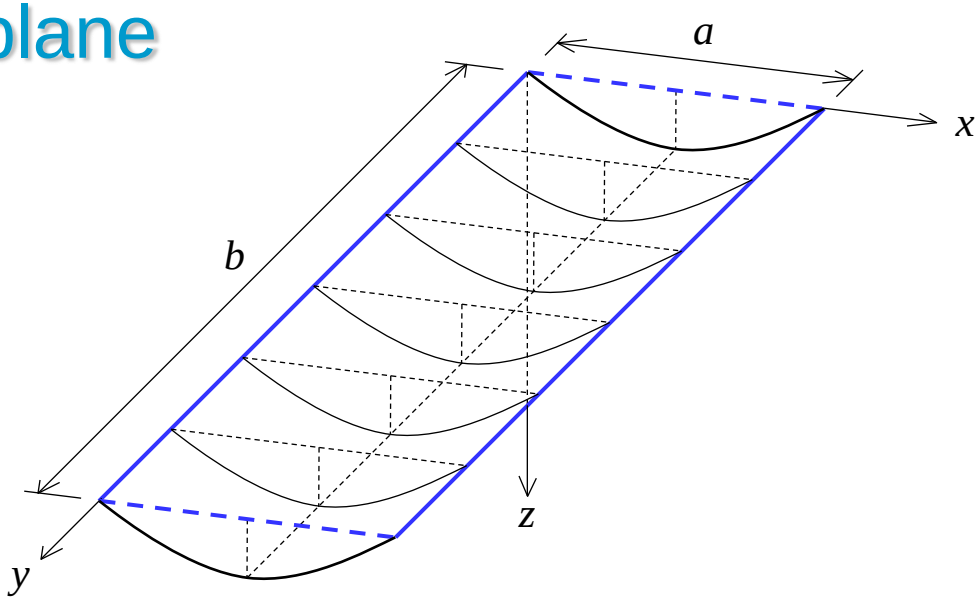
Examples:

- Cylindrical deflection plane

  Biharmonic equation :

$$D\nabla^2\nabla^2 w = p$$

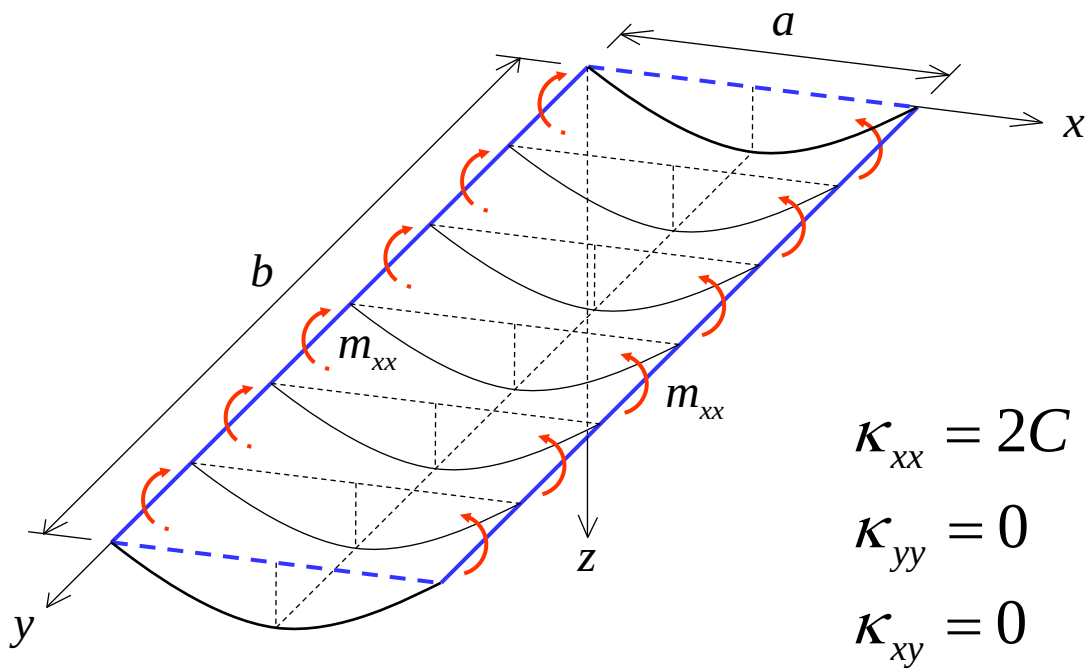


Assume: $w = C\,x\,(a-x)$

$$D\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w = p \quad \Rightarrow \quad p = 0$$

Cylindrical deflection plane

Also for : $w = C\,x\,(a-x)$



$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}$$

$$\kappa_{yy} = -\frac{\partial^2 w}{\partial y^2}$$

$$\kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}$$

$$\kappa_{xx} = 2C$$

$$\kappa_{yy} = 0$$

$$\kappa_{xy} = 0$$

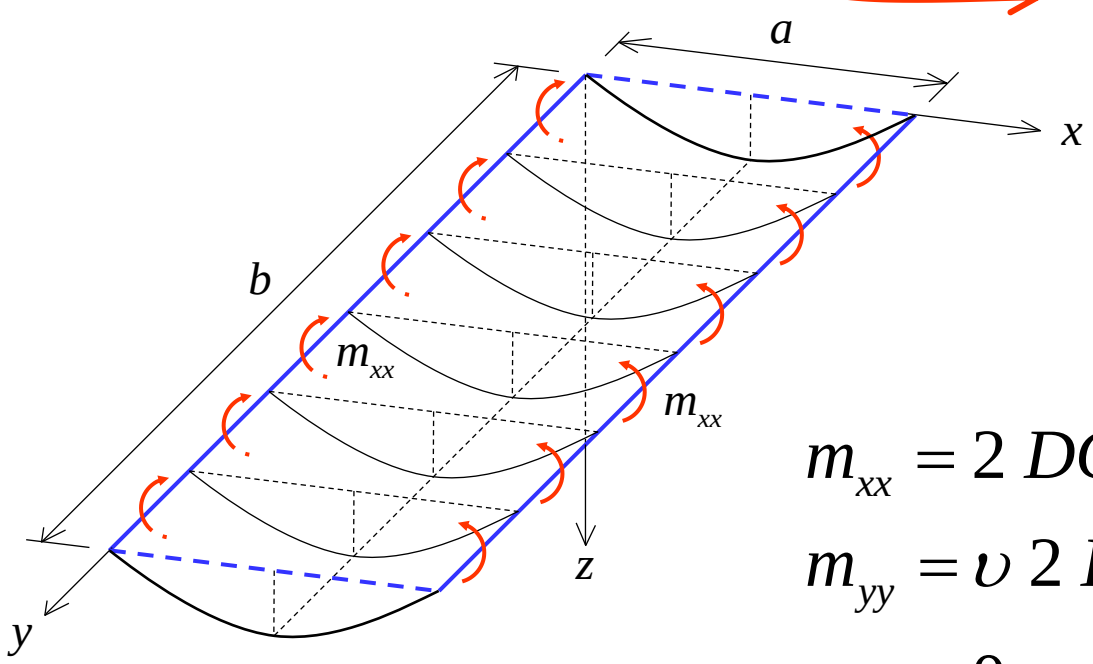
Cylindrical deflection plane

Also with : $w = C\,x\,(a-x)$

$$m_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{yy} = -D\left(\upsilon\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{xy} = -(1-\upsilon)\,D\frac{\partial^2 w}{\partial x \partial y}$$



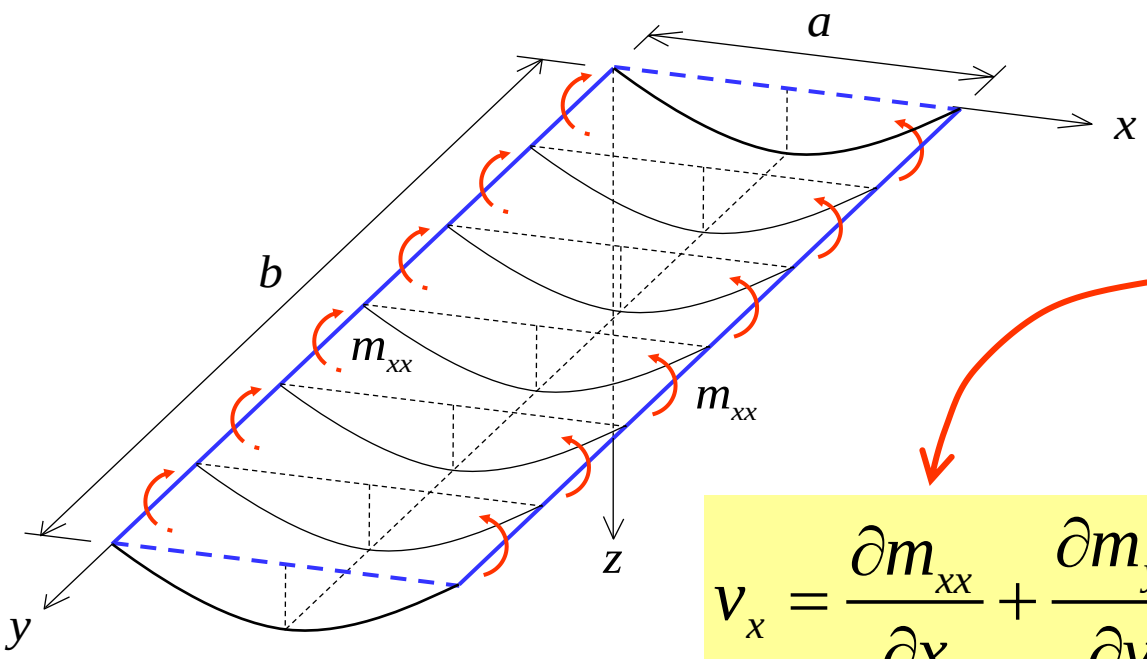
$$m_{xx} = 2\,DC$$

$$m_{yy} = \upsilon\,2\,DC = \upsilon\,m_{xx}$$

$$m_{xy} = 0$$

Cylindrical deflection plane

Also with : $w = C\,x\,(a-x)$



$$m_{xx} = 2\,DC$$

$$m_{yy} = \upsilon\,2\,DC = \upsilon\,m_{xx}$$

$$m_{xy} = 0$$

$$v_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y}$$

$$v_y = \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

$$\Rightarrow \quad v_x = v_y = 0$$

Arbitrary deflection shape

In general : for $w = f(x)$

$$D\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w = p \quad \Rightarrow \quad p = D\frac{d^4}{dx^4} f(x)$$

$$m_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{yy} = -D\left(\upsilon \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

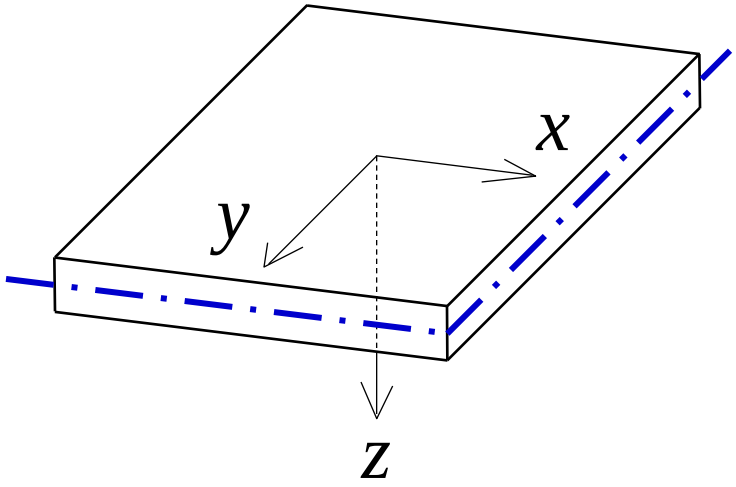
$$m_{xy} = -(1-\upsilon) D \frac{\partial^2 w}{\partial x \partial y}$$

$$\Rightarrow \quad m_{xx} = -D\frac{d^2}{dx^2} f(x)$$

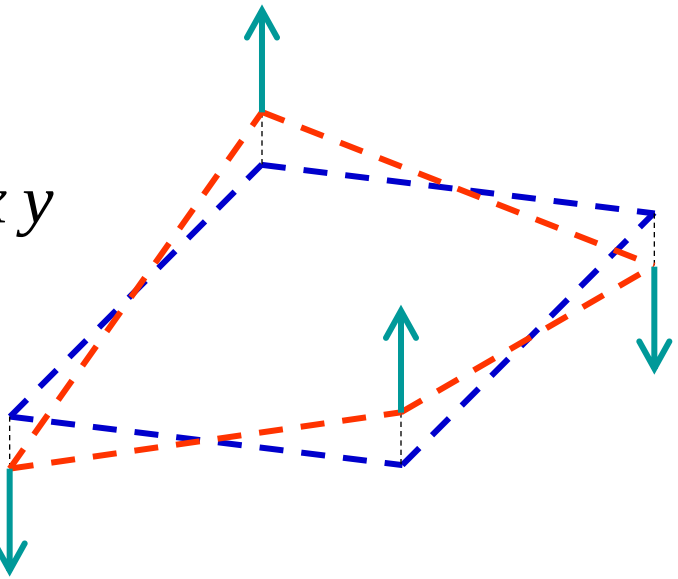
$$m_{yy} = \upsilon\, m_{xx}$$

$$m_{xy} = 0$$

Panel with constant torsion



If : $w = -C\,x\,y$



$$m_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{yy} = -D\left(\upsilon\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

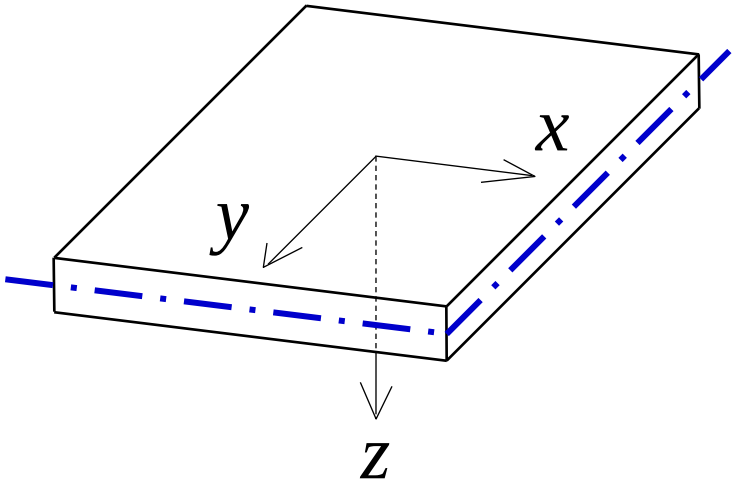
$$m_{xy} = -(1-\upsilon)D\frac{\partial^2 w}{\partial x \partial y}$$

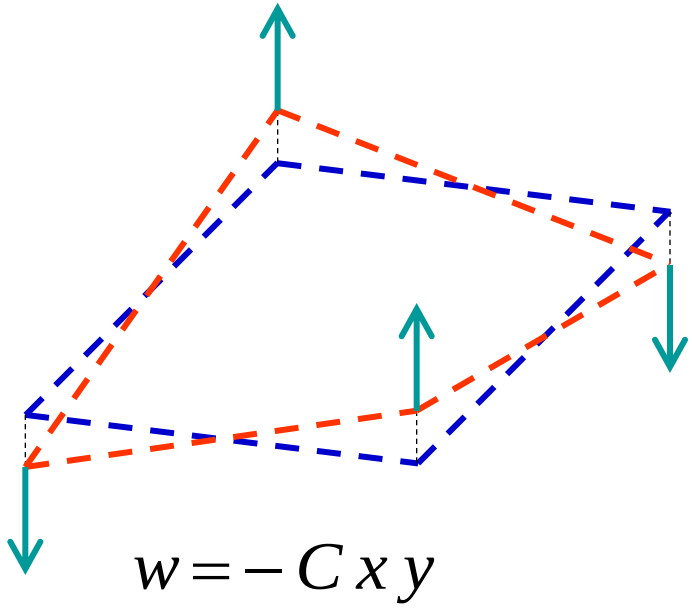
$$m_{xy} = (1-\upsilon)DC$$

$$m_{xx} = 0$$

$$m_{yy} = 0$$

Panel with constant torsion



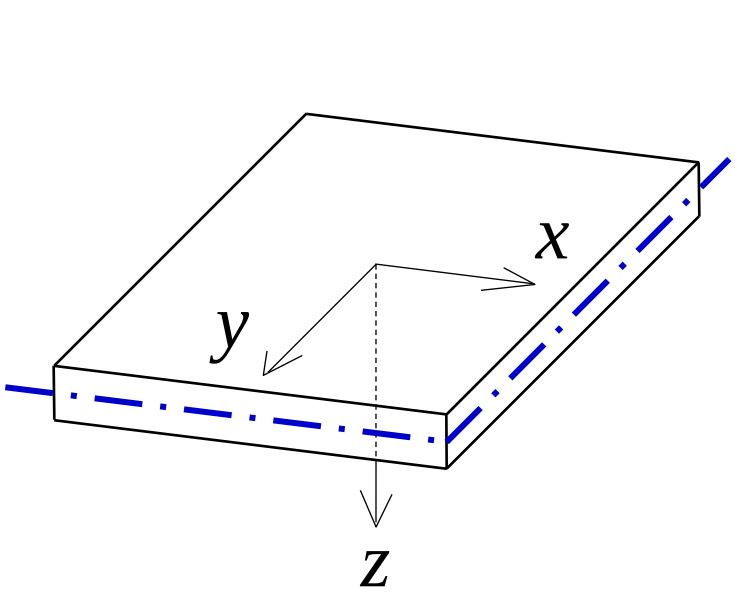


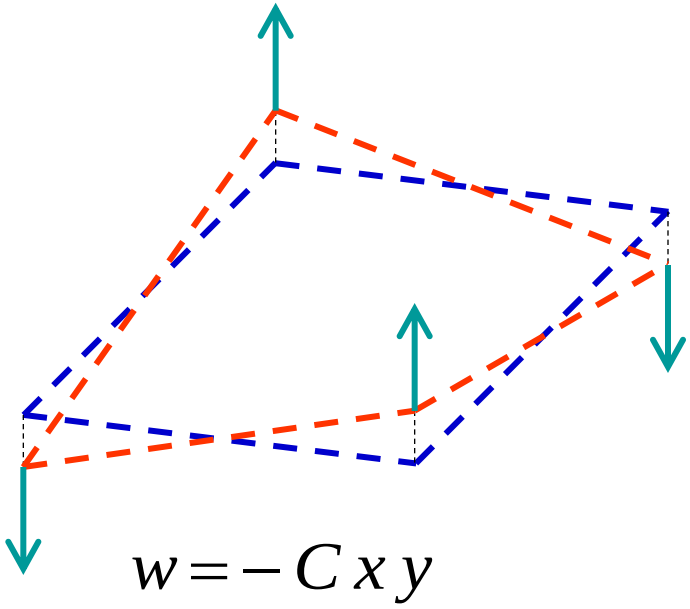
$$m_{xy} = (1-\upsilon) D C$$
$$m_{xx} = 0$$
$$m_{yy} = 0$$

$$-\left(\frac{\partial^2 m_{xx}}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_{yy}}{\partial y^2} \right) = p = 0$$

equilibrium equation

Panel with constant torsion



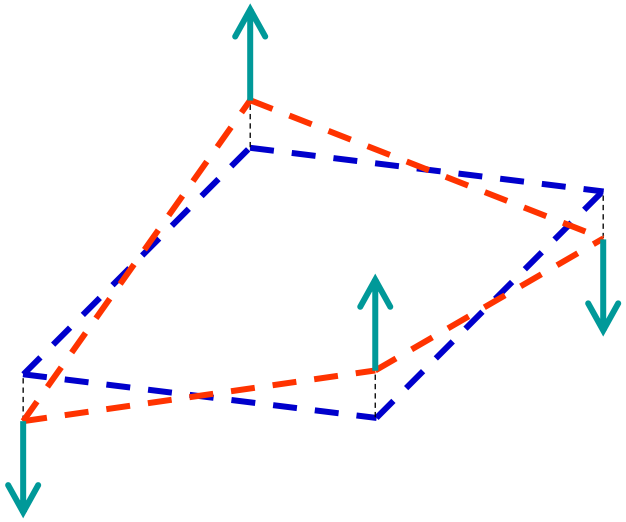


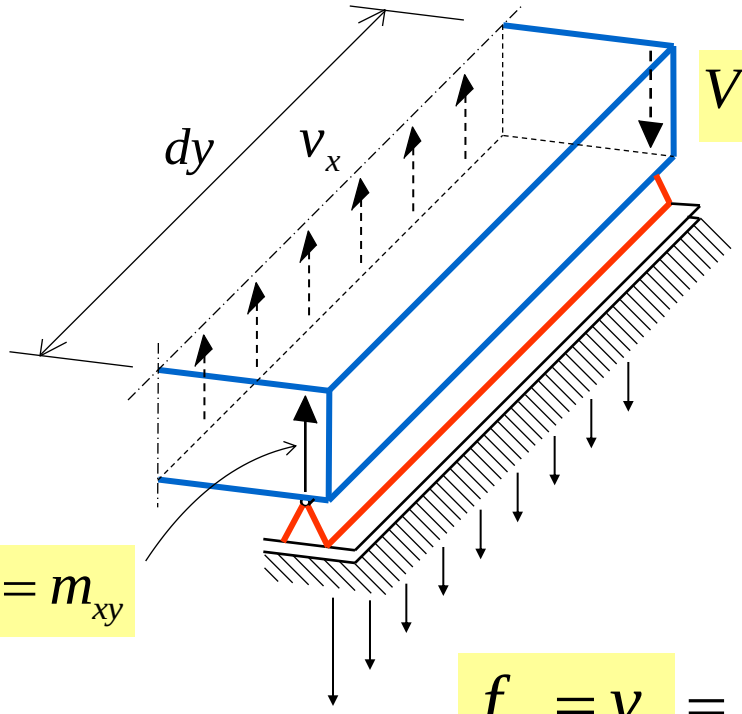
$$m_{xy} = (1-\upsilon) D C$$
$$m_{xx} = 0$$
$$m_{yy} = 0$$

$$v_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y}$$
$$v_y = \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

$$v_x = 0$$
$$v_y = 0$$

Panel with constant torsion

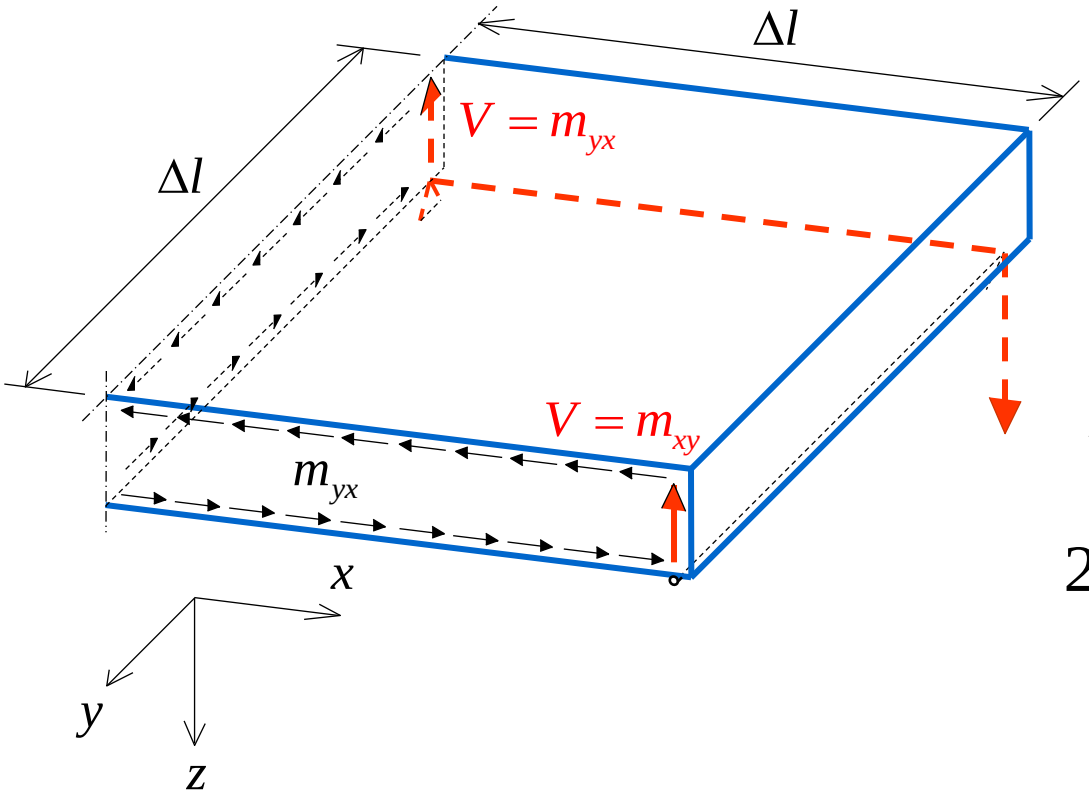




$$m_{xy} = (1-\upsilon) D C$$
$$m_{xx} = 0$$
$$m_{yy} = 0$$
$$v_x = 0$$
$$v_y = 0$$

$$f_w = v_x = 0$$

Panel with constant torsion

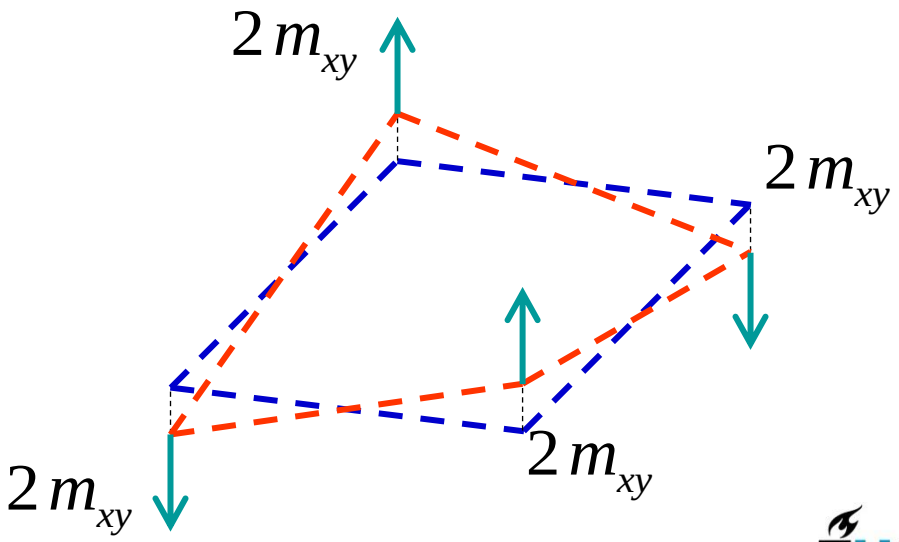


$$m_{xy} = (1-\upsilon) D C$$

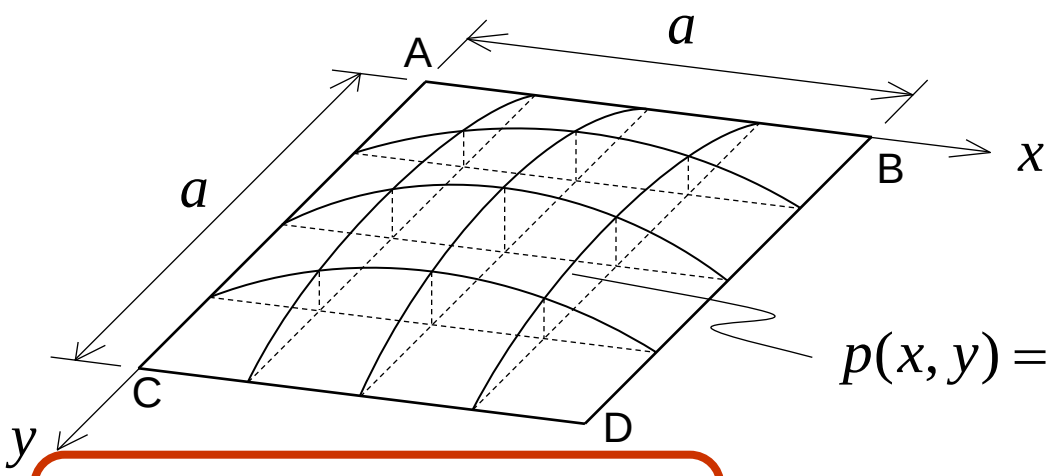
$$m_{xx} = 0$$

$$m_{yy} = 0$$

$$m_{xy} + m_{yx} = 2\,m_{xy}$$



Sinusoidal load on a square plate



$$p(x,y) = p \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

Assume: $w(x,y) = w \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}$ *which conforms to b.c.*:

$$x = 0,\ x = a \rightarrow \begin{cases} w = 0 \\ \frac{\partial^2 w}{\partial x^2} = 0 \end{cases}$$

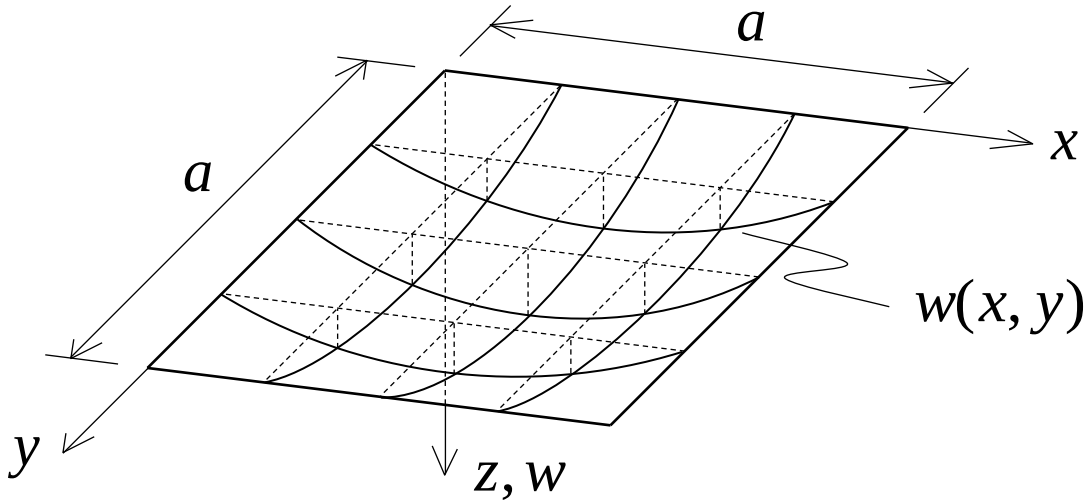
$$y = 0,\ y = a \rightarrow \begin{cases} w = 0 \\ \frac{\partial^2 w}{\partial y^2} = 0 \end{cases}$$

$$D\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w = p$$

Sinusoidal load on a square plate

$$\left[\frac{\pi^4}{a^4}+2\frac{\pi^4}{a^4}+\frac{\pi^4}{a^4}\right]w=\frac{p}{D} \quad \rightarrow \quad w=\frac{pa^4}{4\pi^4 D}$$

$$w(x,y)=w\sin\frac{\pi x}{a}\sin\frac{\pi y}{a} \quad \Rightarrow \quad w(x,y)=\frac{pa^4}{4\pi^4 D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$



$$w_{\max}=\frac{pa^4}{4\pi^4 D}$$

Sinusoidal load on a square plate

$$w(x,y) = \frac{pa^4}{4\pi^4 D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$m_{xx} = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon\frac{\partial^2 w}{\partial y^2}\right)$$

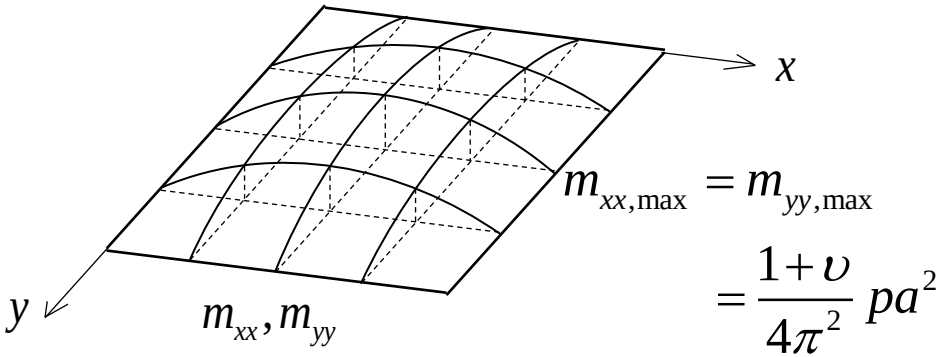
$$m_{yy} = -D\left(\upsilon\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$m_{xy} = -(1-\upsilon)D\frac{\partial^2 w}{\partial x \partial y}$$

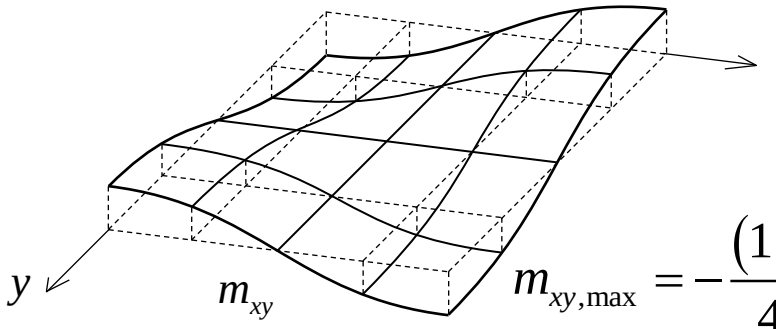
$$m_{xx} = +\frac{1+\upsilon}{4\pi^2}pa^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$m_{yy} = +\frac{1+\upsilon}{4\pi^2}pa^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$m_{xy} = -\frac{1-\upsilon}{4\pi^2}pa^2\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}$$



$$m_{xx,\max} = m_{yy,\max} = \frac{1+\upsilon}{4\pi^2}pa^2$$



$$m_{xy,\max} = -\frac{(1-\upsilon)}{4\pi^2}pa^2$$

Sinusoidal load on a square plate

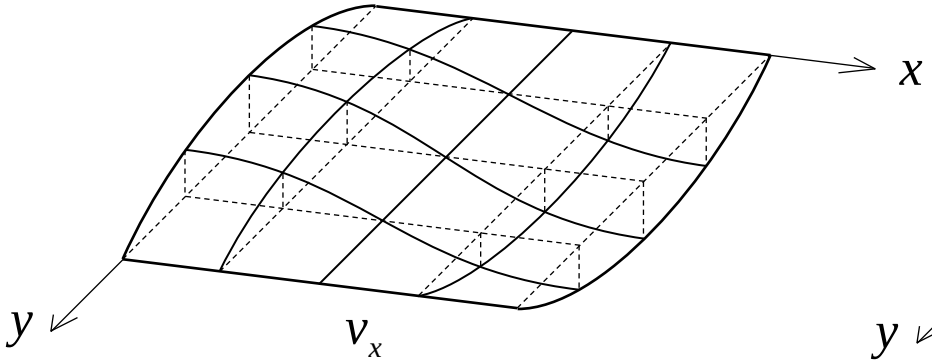
$$m_{xx} = +\frac{1+\upsilon}{4\pi^2} pa^2 \sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

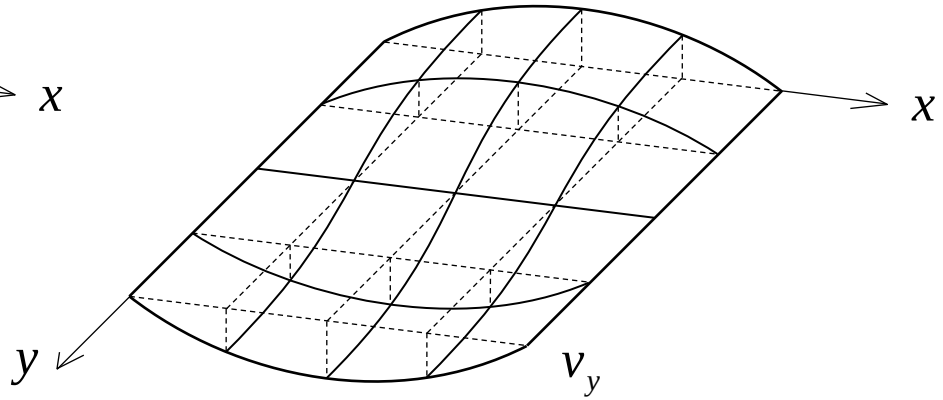
$$m_{yy} = +\frac{1+\upsilon}{4\pi^2} pa^2 \sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$m_{xy} = -\frac{1-\upsilon}{4\pi^2} pa^2 \cos\frac{\pi x}{a}\cos\frac{\pi y}{a}$$

$$v_x = \frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{yx}}{\partial y}$$

$$v_y = \frac{\partial m_{yy}}{\partial y} + \frac{\partial m_{xy}}{\partial x}$$

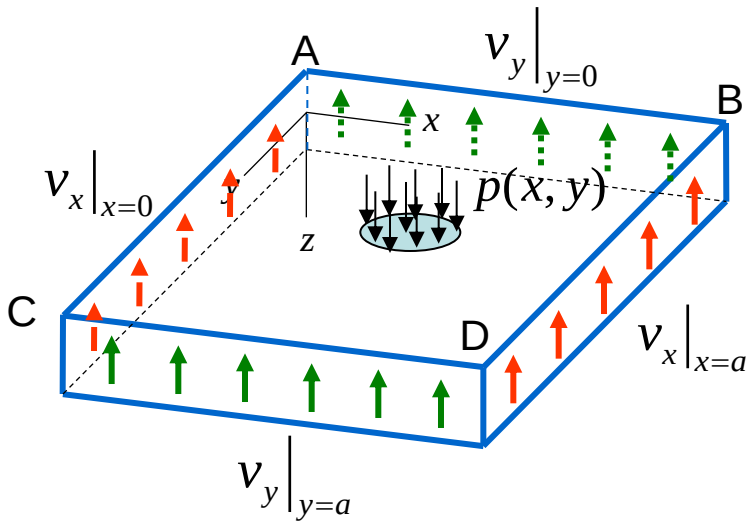




$$v_x = \frac{1}{2\pi} p\,a\,\cos\frac{\pi x}{a}\,\sin\frac{\pi y}{a}$$

$$v_y = \frac{1}{2\pi} p\,a\,\sin\frac{\pi x}{a}\,\cos\frac{\pi y}{a}$$

Sinusoidal load on a square plate



$$p(x,y) = p\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}$$

$$v_x\big|_{x=0} = \frac{1}{2\pi}\,p\,a\sin\frac{\pi y}{a}$$

$$v_x\big|_{x=a} = -\frac{1}{2\pi}\,p\,a\sin\frac{\pi y}{a}$$

$$v_y\big|_{y=0} = \frac{1}{2\pi}\,p\,a\sin\frac{\pi x}{a}$$

$$v_y\big|_{y=a} = -\frac{1}{2\pi}\,p\,a\sin\frac{\pi x}{a}$$

Total shear force 4S along the edges:

$$4S = \int_0^a \left(\left|v_x\big|_{x=0}\right| + \left|v_x\big|_{x=a}\right|\right)dy + \int_0^a \left(\left|v_y\big|_{y=0}\right| + \left|v_y\big|_{y=a}\right|\right)dx = \frac{4}{\pi^2}\,pa^2$$

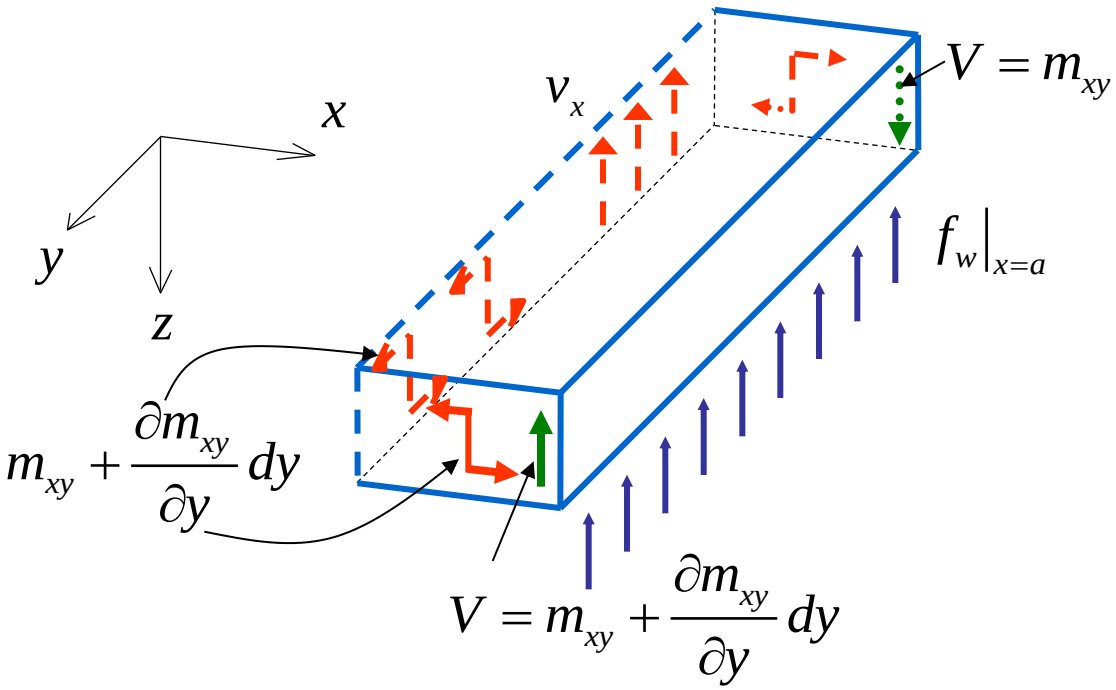
Total load P applied on the top surface of the plate:

$$P = \int_0^a\int_0^a p(x,y)\,dxdy = p\int_0^a \sin\frac{\pi x}{a}\,dx\int_0^a \sin\frac{\pi y}{a}\,dy = p\,\frac{2a}{\pi}\,\frac{2a}{\pi} = \frac{4}{\pi^2}\,pa^2$$

$$4S = P$$

Sinusoidal load on a square plate

- support reactions @x=a



$$v_x = \frac{1}{2\pi} p a \cos\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

$$\frac{\partial m_{xy}}{\partial y} = \frac{1-\upsilon}{4\pi^2} p a^2 \cos\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

$$v_x\big|_{x=a} = -\frac{1}{2\pi} p a \sin\frac{\pi y}{a}$$

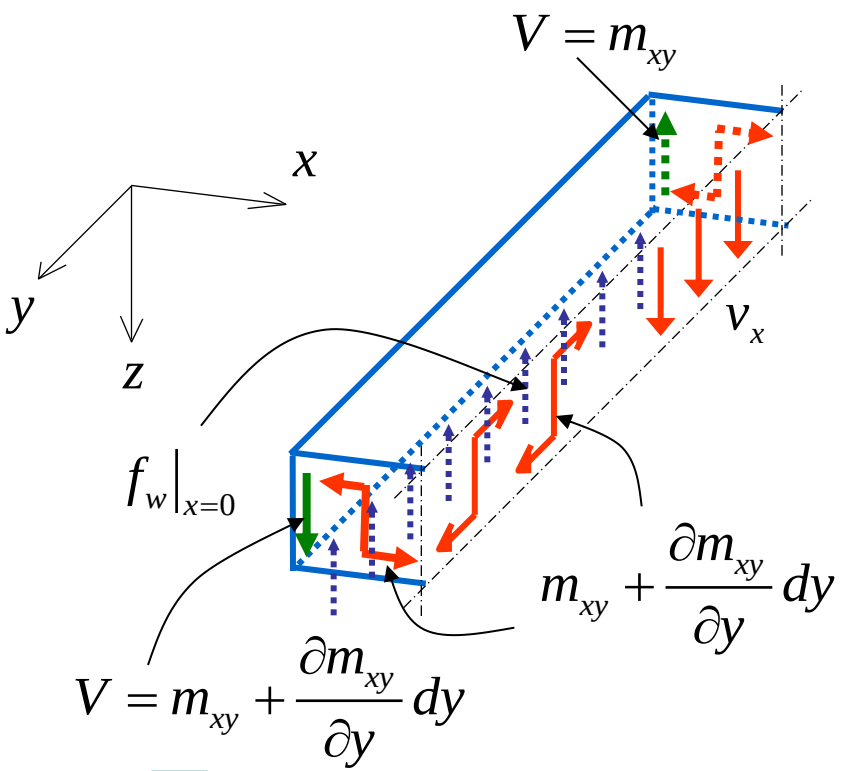
$$\left.\frac{\partial m_{xy}}{\partial y}\right|_{x=a} = -\frac{1-\upsilon}{4\pi} p a \sin\frac{\pi y}{a}$$

$$f_w = \left(v_x + \frac{\partial m_{xy}}{\partial y}\right)_{x=a} = -\frac{3-\upsilon}{4\pi} p a \sin\frac{\pi y}{a}$$

points upwards

Sinusoidal load on a square plate

- support reactions @x=0



$$v_x = \frac{1}{2\pi} p a \cos\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

$$\frac{\partial m_{xy}}{\partial y} = \frac{1-\upsilon}{4\pi^2} p a^2 \cos\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

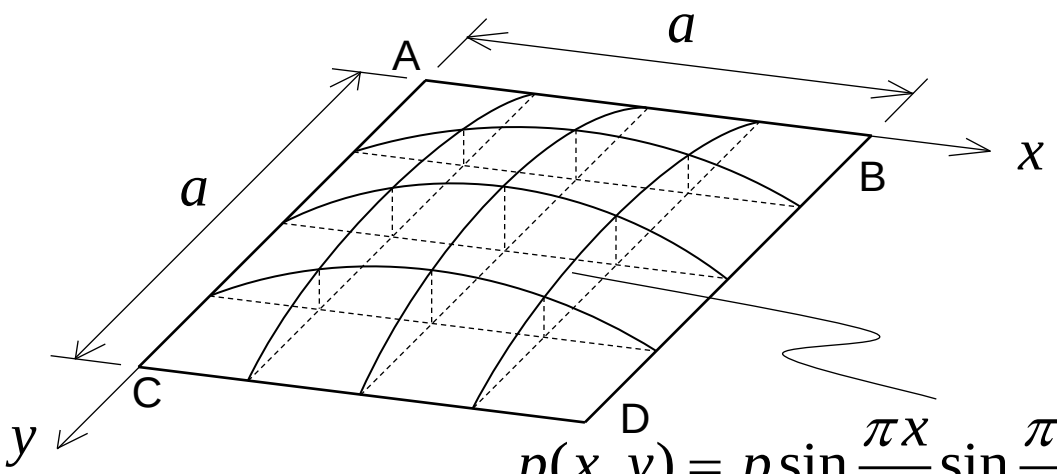
$$v_x\big|_{x=0} = \frac{1}{2\pi} p a \sin\frac{\pi y}{a}$$

$$\left.\frac{\partial m_{xy}}{\partial y}\right|_{x=0} = \frac{1-\upsilon}{4\pi} p a \sin\frac{\pi y}{a}$$

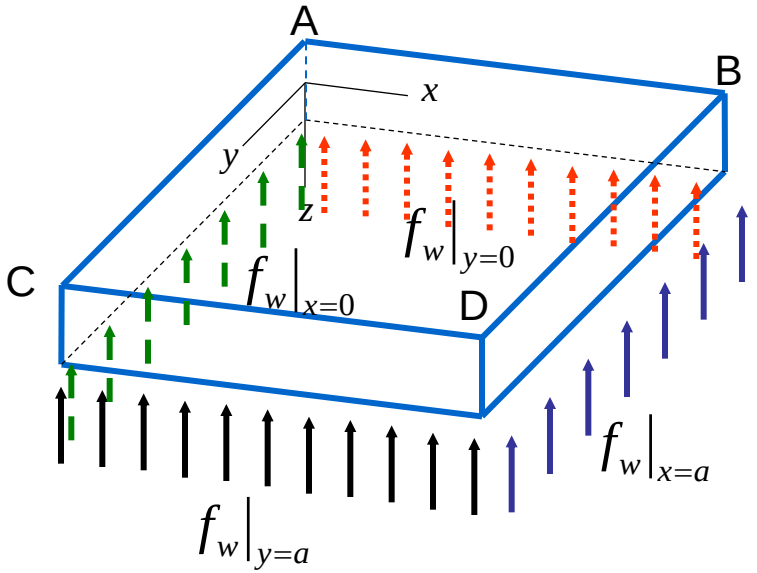
$$f_w = -\left(v_x + \frac{\partial m_{xy}}{\partial y}\right)_{x=0} = -\frac{3-\upsilon}{4\pi} p a \sin\frac{\pi y}{a}$$

points upwards

Sinusoidal load on a square plate



$$p(x,y) = p \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}$$



- total support reactions :

$$\int_0^a \left(f_w|_{x=0} + f_w|_{x=a} \right) dy + \int_0^a \left(f_w|_{y=0} + f_w|_{y=a} \right) dx =$$

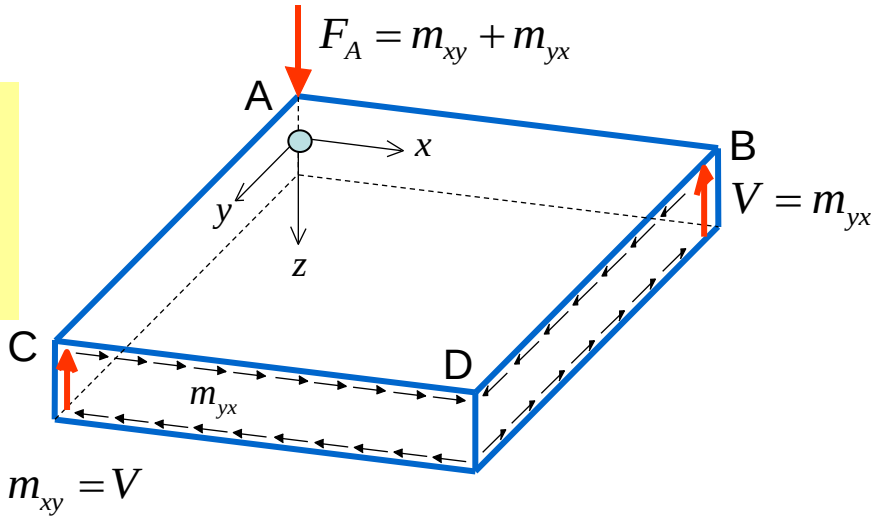
points upwards

$$2\left| \frac{3-\nu}{4\pi}\, pa \int_0^a \sin\frac{\pi y}{a}\, dy \right| + 2\left| \frac{3-\nu}{4\pi}\, pa \int_0^a \sin\frac{\pi x}{a}\, dx \right| = \left(\frac{6-2\nu}{\pi^2} \right) pa^2 = 4R$$

$$|4R| = \left(\frac{6-2\nu}{\pi^2}\right) pa^2 > 4\frac{pa^2}{\pi^2}$$

total sinusoidal load

$$m_{xy} = -\left(\frac{1-\nu}{4\pi^2}\, pa^2 \cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right)$$



at a corner A:

$$m_{xy}\Big|_{\substack{x=0\\y=0}} = m_{yx}\Big|_{\substack{x=0\\y=0}} = -\frac{1-\nu}{4\pi^2}\, pa^2 = V$$

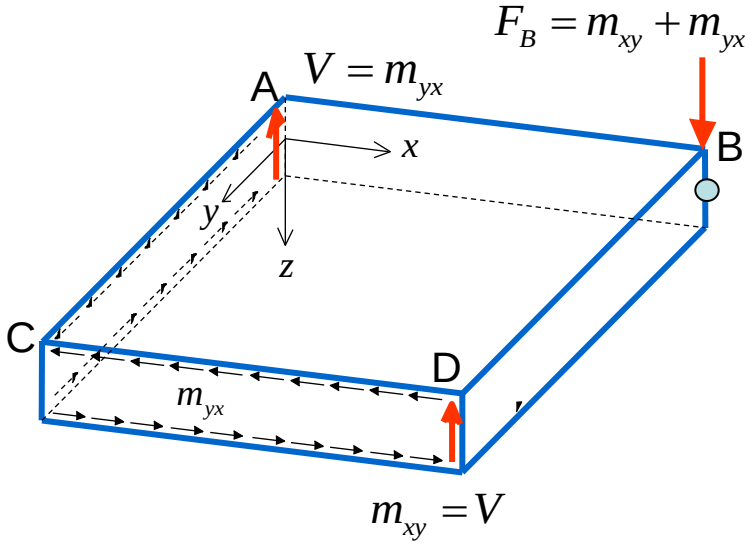
$$F_A = |m_{xy} + m_{yx}| = 2V = \frac{1-\nu}{2\pi^2}\, pa^2$$

points downwards

$$|4R| = \left(\frac{6-2\nu}{\pi^2}\right) pa^2 > 4\frac{pa^2}{\pi^2}$$

total sinusoidal load

$$m_{xy} = -\left(\frac{1-\nu}{4\pi^2}\, pa^2 \cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right)$$



at a corner B:

$$m_{xy}\Big|_{\substack{x=a\\y=0}} = m_{yx}\Big|_{\substack{x=a\\y=0}} = \frac{1-\nu}{4\pi^2}\, pa^2 = V$$

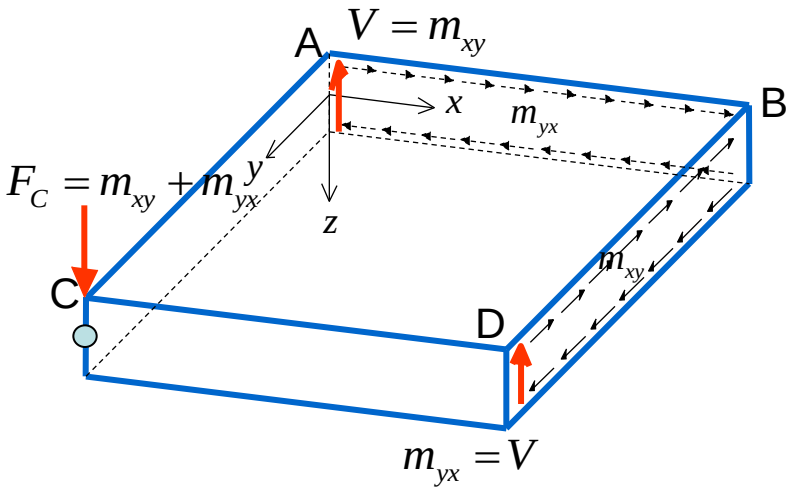
$$F_B = |m_{xy} + m_{yx}| = 2V = \frac{1-\nu}{2\pi^2}\, pa^2$$

points downwards

$$|4R| = \left(\frac{6-2\nu}{\pi^2}\right) pa^2 > 4\frac{pa^2}{\pi^2}$$

total sinusoidal load

$$m_{xy} = -\left(\frac{1-\nu}{4\pi^2}\, pa^2 \cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right)$$



at a corner C:

$$m_{xy}\Big|_{\substack{x=0\\y=a}} = m_{yx}\Big|_{\substack{x=0\\y=a}} = \frac{1-\nu}{4\pi^2}\, pa^2 = V$$

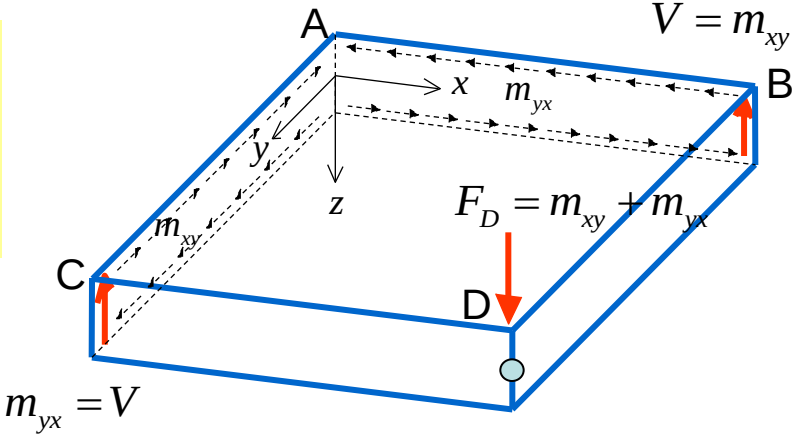
$$F_C = |m_{xy} + m_{yx}| = 2V = \frac{1-\nu}{2\pi^2}\, pa^2$$

points downwards

$$|4R| = \left(\frac{6-2\nu}{\pi^2}\right) pa^2 > 4\frac{pa^2}{\pi^2}$$

total sinusoidal load

$$m_{xy} = -\left(\frac{1-\nu}{4\pi^2}\, pa^2 \cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\right)$$



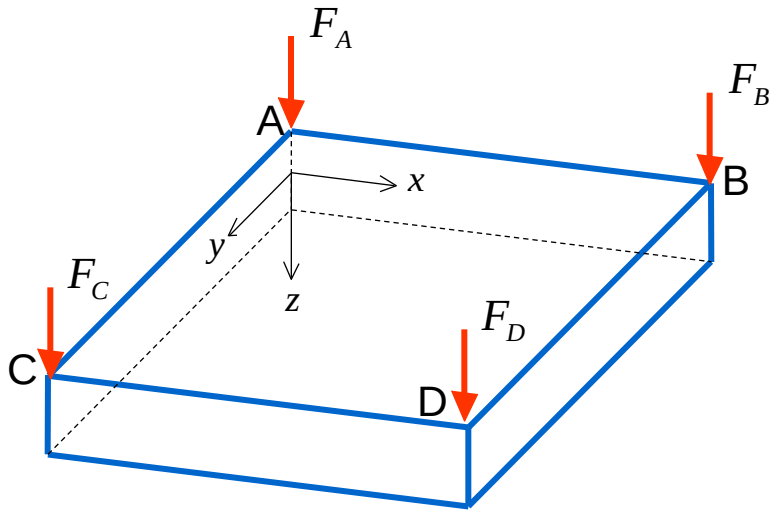
at a corner D:

$$m_{xy}\Big|_{\substack{x=a\\y=a}} = m_{yx}\Big|_{\substack{x=a\\y=a}} = -\frac{1-\nu}{4\pi^2}\, pa^2 = V$$

$$F_D = |m_{xy} + m_{yx}| = 2V = \frac{1-\nu}{2\pi^2}\, pa^2$$

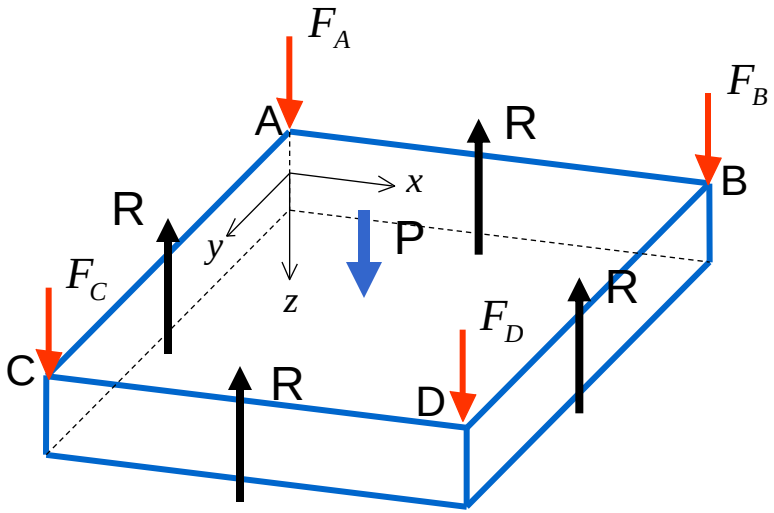
points downwards

Corner reaction forces



➡ totally :

$$4F = F_A + F_B + F_C + F_D = \frac{2-2\nu}{\pi^2} pa^2$$



Total vertical force balance:

$$4R + 4F = -\frac{6-2\nu}{\pi^2} pa^2 + \frac{2-2\nu}{\pi^2} pa^2 = -\frac{4}{\pi^2} pa^2 = -P$$