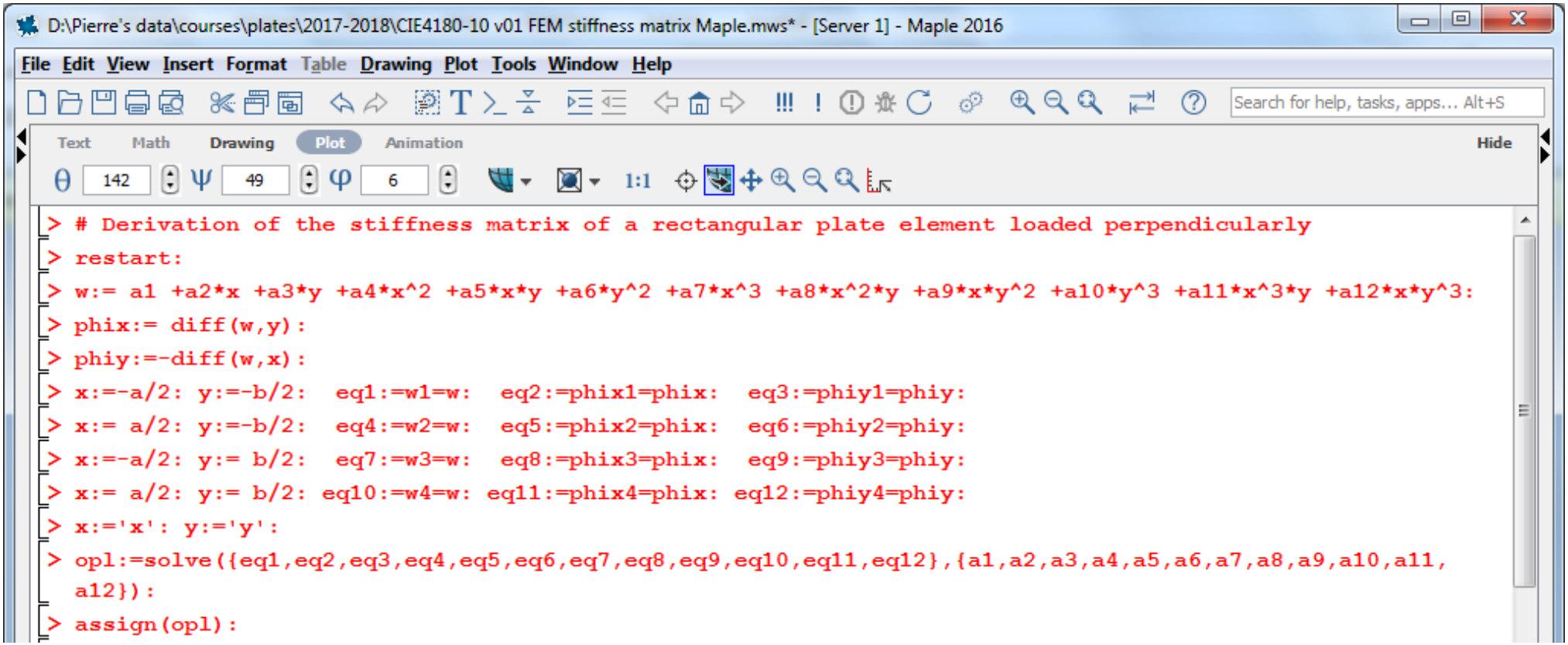
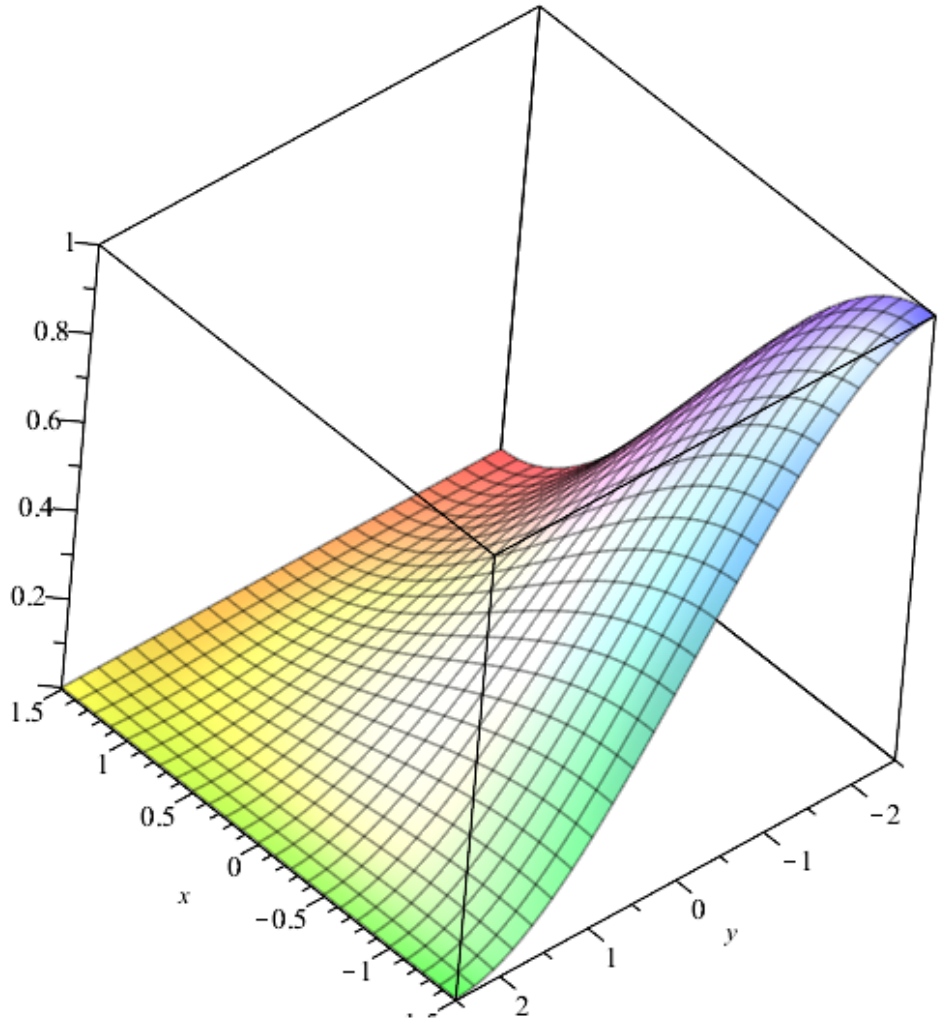
Finite Element Method

implicit interpolation



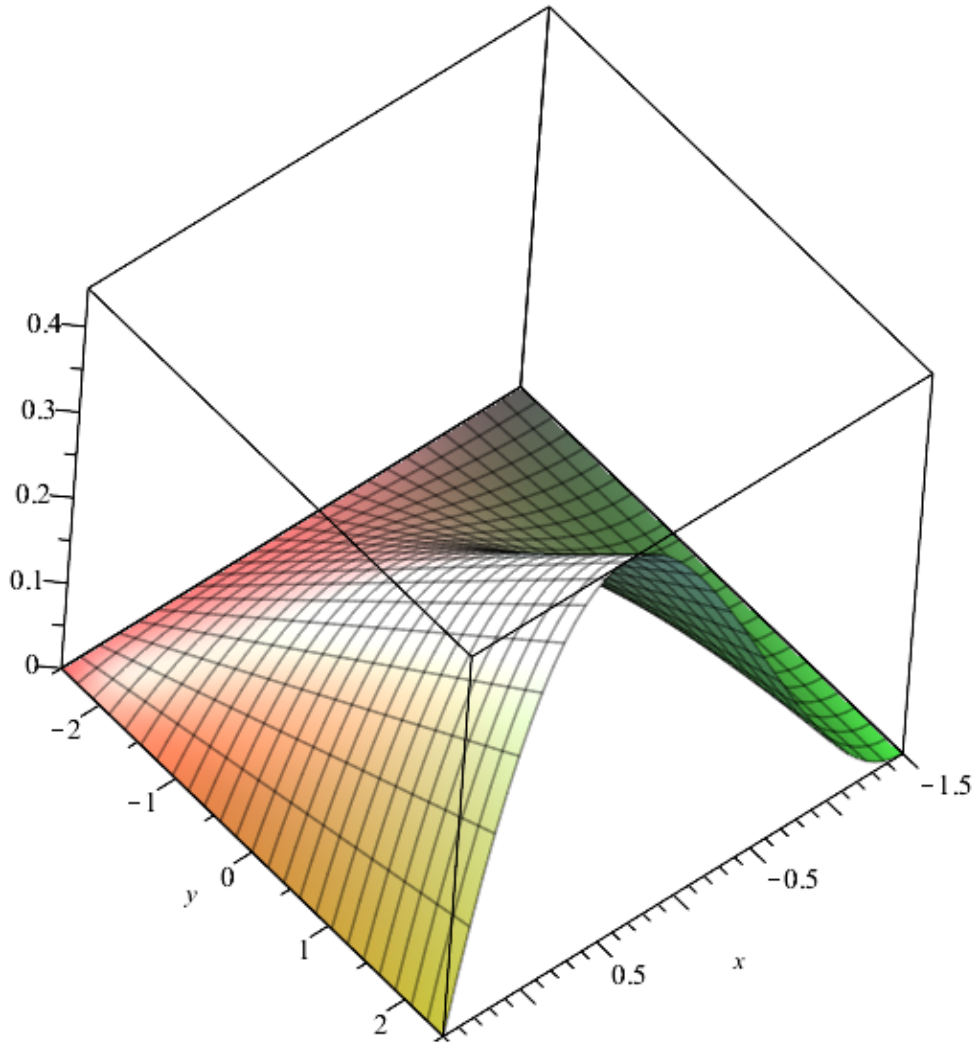
shape function 1

```
> a:=3: b:=5: w1:=1: w2:=0: w3:=0: w4:=0: phix1:=0: phix2:=0: phix3:=0: phix4:=0: phiy1:=0: phiy2:=0:
  phiy3:=0: phiy4:=0:
> plot3d(w,x=-a/2..a/2,y=-b/2..b/2);
```



shape function 12

```
> a:=3: b:=5: w1:=0: w2:=0: w3:=0: w4:=0: phix1:=0: phix2:=0: phix3:=0: phix4:=0: phiy1:=0: phiy2:=0:
  phiy3:=0: phiy4:=1:
> plot3d(w,x=-a/2..a/2,y=-b/2..b/2);
```



Stiffness matrix derived from the thin plate theory

```
> a:='a': b:='b': w1:='w1': w2:='w2': w3:='w3': w4:='w4': phix1:='phix1': phix2:='phix2': phix3:=
  'phix3': phix4:='phix4': phiy1:='phiy1': phiy2:='phiy2': phiy3:='phiy3': phiy4:='phiy4':
>
> kappaxx:=-diff(w,x,x):
> kappayy:=-diff(w,y,y):
> rhoxy:=-2*diff(w,x,y):
> mxx:=D*(kappaxx+nu*kappayy):
> myy:=D*(kappayy+nu*kappaxx):
> mxy:=1/2*(1-nu)*D*rhoxy:
> Es:=1/2*kappaxx*mxx +1/2*kappayy*myy +1/2*rhoxy*mxy:
> Ess:=int(int(Es,x=-a/2..a/2),y=-b/2..b/2):
> K1_1:=collect(expand(diff(Ess,w1,w1)),D);
```

$$K1_1 := \left(\frac{14}{5\,a\,b} - \frac{4}{5}\,\frac{\nu}{a\,b} + \frac{4\,b}{a^3} + \frac{4\,a}{b^3} \right) \mathrm{D}$$

```
> K1_2:=collect(expand(diff(Ess,w1,phix1)),D);
```

$$K1_2 := \left(\frac{1}{5\,a} + \frac{4}{5}\,\frac{\nu}{a} + \frac{2\,a}{b^2} \right) \mathrm{D}$$

```
> # et cetera.
```

$$\begin{bmatrix} F_1 \\ M_{x1} \\ M_{y1} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ M_{y4} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & & & & & & & & & \\ & . & & & & & & & & & \\ & & . & & & & & & & & \\ & & & . & & & & & & & \\ & & & & . & & & & & & \\ & & & & & . & & & & & \\ & & & & & & . & & & & \\ & & & & & & & . & & & \\ & & & & & & & & . & & \\ & & & & & & & & & . & \\ & & & & & & & & & & K_{1212} \end{bmatrix} \begin{bmatrix} w_1 \\ \varphi_{x1} \\ \varphi_{y1} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ \varphi_{y4} \end{bmatrix}$$

$$K_{11} = \frac{\partial^2 E_{ss}}{\partial w_1^2}$$

$$K_{12} = \frac{\partial^2 E_{ss}}{\partial w_1 \partial \varphi_{x1}}$$

et cetera

Stiffness matrix derived from 3D elasticity

```
> # Again but this time starting from "needle remains straight".
> E:=D/t^3*(12*(1-nu^2)):
> G:=E/(2*(1+nu)):
> epsilonxx:=z*kappaxx:
> epsilonyy:=z*kappayy:
> gammaxy:=z*rhoxy:
> sigmaxx:=E/(1-nu^2)*(epsilonxx+nu*epsilonyy):
> sigmayy:=E/(1-nu^2)*(epsilonyy+nu*epsilonxx):
> sigmaxy:=G*gammaxy:
> Es:=1/2*sigmaxx*epsilonxx +1/2*sigmayy*epsilonyy +1/2*sigmaxy*gammaxy:
> Ess:=int(int(int(Es,z=-t/2..t/2),x=-a/2..a/2),y=-b/2..b/2):
> K1_1:=collect(expand(simplify(diff(Ess,w1,w1))),D);
```

$$\left(\frac{14}{5\,a\,b}-\frac{4}{5}\,\frac{\nu}{a\,b}+\frac{4\,b}{a^3}+\frac{4\,a}{b^3}\right)\mathrm{D}$$

```
> K1_2:=collect(expand(simplify(diff(Ess,w1,phix1))),D);
```

$$\left(\frac{1}{5\,a}+\frac{4}{5}\,\frac{\nu}{a}+\frac{2\,a}{b^2}\right)\mathrm{D}$$

Plotting the moments etc. in an element

```
> # functions of the moments and shear forces
> restart:
> w:= a1 +a2*x +a3*y +a4*x^2 +a5*x*y +a6*y^2 +a7*x^3 +a8*x^2*y +a9*x*y^2 +a10*y^3 +a11*x^3*y
  +a12*x*y^3:
> kappaxx:=-diff(w,x,x):
> kappayy:=-diff(w,y,y):
> rhoxy:=-2*diff(w,x,y):
> mxx:=D*(kappaxx+nu*kappayy):
> myy:=D*(kappayy+nu*kappaxx):
> mxy:=1/2*(1-nu)*D*rhoxy:
> vx:=diff(mxx,x)+diff(mxy,y):
> vy:=diff(myy,y)+diff(mxy,x):
> collect(mxx/D,{x,y});          # mxx is linear in x and y plus it has x*y.
```

$$((-6\,a12\,\nu - 6\,a11)\,y - 2\,a9\,\nu - 6\,a7)\,x + (-6\,a10\,\nu - 2\,a8)\,y - 2\,a6\,\nu - 2\,a4$$

```
> collect(myy/D,{x,y});          # myy is linear in x and y plus it has x*y.
```

$$((-6\,a11\,\nu - 6\,a12)\,y - 6\,a7\,\nu - 2\,a9)\,x + (-2\,a8\,\nu - 6\,a10)\,y - 2\,a4\,\nu - 2\,a6$$

```
> collect(mxy/D,{x,y});          # mxy is parabolic in x and y but it has no x*y.
```

$$-3\,(1-\nu)\,a11\,x^2 - 2\,(1-\nu)\,a8\,x - 3\,(1-\nu)\,a12\,y^2 - 2\,(1-\nu)\,a9\,y - (1-\nu)\,a5$$

```
> collect(simplify(vx/D),{x,y}); # vx is linear in y and constant in x.
```

$$(-6\,a11 - 6\,a12)\,y - 6\,a7 - 2\,a9$$

```
> collect(simplify(vy/D),{x,y}); # vy is linear in x and constant in y.
```

$$(-6\,a11 - 6\,a12)\,x - 2\,a8 - 6\,a10$$