CT 5143 Stresses in Shells

Exam, 24 January 2005, 14:00 – 17:00 hours

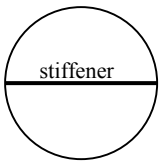
Question 1 (3 points)

Scientists want to observe the fauna on the ocean-floor. They decide to have built a observation capsule in the shape of a spherical shell. It has a radius *a,* and consists of plate material of constant thickness *t*. The modulus of elasticity is *E*. Poisson's ratio can be set to zero. The area of the observation window is small if compared to the area of the sphere and can be neglected.
The capsule is sufficiently ballasted to have sunk down. The depth *d* of the ocean is large if compared tot the radius of the sphere, so it can be assumed that the sphere is loaded by a constant radial pressure *p*. The specific mass of water is ρ and the acceleration of gravity *g*. Because of symmetry over all planes through the sphere centre the membrane forces N_x and N_y are equal.

a) Derive the theoretical critical buckling strength $\sigma_{cr} = \frac{1}{3}\sqrt{3}\,Et/a$ and the buckling length $l_{cr} = \pi/\sqrt[4]{3}\,\sqrt{at}$. (Suppose a buckled shape with the same buckling length in both *x*-direction and in *y*-direction).
b) Why is a large safety factor γ_b for shell buckling needed?
c) Show that the required thickness to avoid buckling can be computed from
$$t^2 = \frac{\sqrt{3}}{2}\frac{\gamma_b \rho g d a^2}{E}$$
d) Design the shell thickness for the following data. Depth of ocean *d* = 400 m. The specific mass of water ρ = 1000 kg/m^3 and the acceleration of gravity *g* = 10 m/s^2. Radius *a* = 1.5 m. Yield strength σ_y = 355 MPa, Elasticity modulus *E* = 2.1 10^5 MPa. Safety for strength: $\gamma_s = 1.6$. Safety for buckling: γ_b = 7.

Question 2 (4 points)

Consider again the problem of question 1. Now an alternative sphere is studied, which has an additional horizontal ring-stiffener halfway the depth of the sphere. This shell is axisymmetric with a vertical axis of rotation. The stiffener is made of the same material as the shell, has a cross-section area *A*. It will disturb the membrane stress state in the shell.

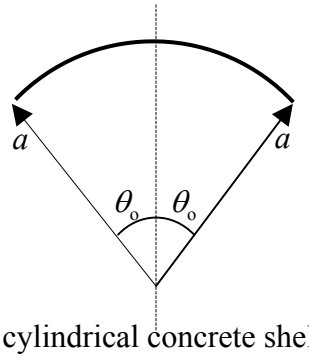


a) Prove that the formula for the meridian bending stress (in *x*-direction) at the connection of the ring to the shell can be written as:
$$\sigma_{xx} = \frac{\sqrt{3}}{2 + \dfrac{4t}{\mu A}} \cdot \frac{a}{t} p$$

where μ is defined by $\mu^{-4} = a^2 t^2 / 3$. (A possible approach: first determine f_r between the ring and the shell and after that derive t_x from f_r).

b) Compute the total stresses σ_{xx} (sum of membrane and bending) at both faces of the shell.
Choose for the stiffener cross-section area $A = 4t / \mu$ and for the shell thickness t = 0.02 m, and use for the rest the data mentioned in question 1.

Question 3 (3 points)



A roof is built as a cylindrical concrete shell as shown in the figure. The span is l = 40 m, the radius a = 16 m, thickness t = 0.08 m and the opening angle $\theta_o = 30^\circ$. Poisson's ratio is zero. The shell is simply supported. At both ends a diaphragm (tympan) occurs.

The shell is loaded by dead weight, roof covering and snow, which together have the value p = 3,5 kN/m^2. This load is in pure vertical direction. We consider the cross-section of the shell halfway the span.

a) Which boundary condition must be satisfied at the straight edges of the shell?
b) Compute the stress σ_{xx} at the edge of the shell in the cross-section mid-span at both faces of the shell (membrane and bending stresses summed up). Use formulas already derived in the lecture notes.
c) Comment on the stresses you have calculated:
   - Show that it is hard to reinforce the shell (design strength reinforcement 435 N/mm²).
   - Why will it not be helpful to enlarge the thickness?
   - Which structural change(s) would you propose?

Answer to question 1a

The general buckling equation is (p. 159)

$$\frac{Et^3}{12}\left(\frac{1}{A^2}+\frac{1}{B^2}\right)^4+Et\left(\frac{1}{A^2R_y}+\frac{1}{B^2R_x}\right)^2=\left(\frac{N_x}{A^2}+\frac{N_y}{B^2}\right)\left(\frac{1}{A^2}+\frac{1}{B^2}\right)^2.$$

In this question we use

$R_x = R_y = a$

$B = A$

$N_x = N_y = N$. (The membrane forces are equal).

Therefore, the buckling equation becomes

$$\frac{Et^3}{12}\left(\frac{2}{A^2}\right)^4+\frac{Et}{a^2}\left(\frac{2}{A^2}\right)^2=\left(\frac{2N}{A^2}\right)\left(\frac{2}{A^2}\right)^2.$$

This can be simplified to

$$\frac{4}{3}\frac{Et^3}{A^8}+4\frac{Et}{a^2A^4}=8\frac{N}{A^6}$$

$$\frac{1}{6}\frac{Et^3}{A^2}+\frac{1}{2}\frac{EtA^2}{a^2}=N. \qquad (*1)$$

We write this in the form

$$N=Y\left(X+\frac{1}{X}\right).$$

Herein is

$$Y=\frac{Et^2}{a\sqrt{12}} \text{ and } X=\frac{\sqrt{12}}{6}\frac{at}{A^2}.$$

N is minimal for $X = 1$, so

$$N_{cr}=2Y=\frac{2}{\sqrt{12}}\frac{Et^2}{a}=\tfrac{1}{3}\sqrt{3}\frac{Et^2}{a}$$

$$\sigma_{cr}=\frac{N_{cr}}{t}=\tfrac{1}{3}\sqrt{3}\frac{Et}{a} \qquad (*2)$$

which was to be proved.

<u>Buckling Length</u>

$$X=\frac{\sqrt{12}}{6}\frac{at}{A^2}=1, \text{ so } A^2=\frac{1}{\sqrt{3}}at \text{ and } A=\frac{1}{\sqrt[4]{3}}\sqrt{at}$$

$$l_{cr} = \pi A = \frac{\pi}{\sqrt[4]{3}}\sqrt{at} = 2.39\sqrt{at}$$

Alternative derivation

We start from equation (*1).

$$\frac{1}{6}\frac{Et^3}{A^2} + \frac{1}{2}\frac{Et\,A^2}{a^2} = N$$

The minimum value of N with respect to the buckling length A is found by differentiation.

$$0 = \frac{\partial N}{\partial A} = \frac{Et^3}{6}\frac{-2}{A^3} + \frac{Et}{a^2}A$$

From this we calculate

$$A^2 = \frac{at}{\sqrt{3}}\,.$$

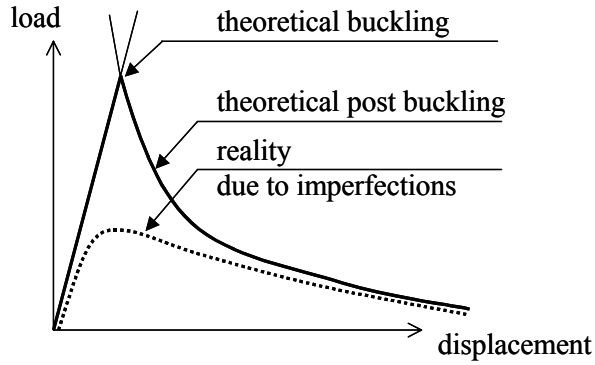
Substitution in (*1) gives

$$\sigma_{cr} = \frac{N_{cr}}{t} = \frac{1}{6}\frac{Et^2}{a^2} + \frac{1}{2}\frac{EA^2}{a^2} = \frac{1}{6}\frac{Et^2\sqrt{3}}{at} + \frac{1}{2}\frac{Eat}{a^2\sqrt{3}} = \frac{1}{6}\frac{Et\sqrt{3}}{a} + \frac{1}{2}\frac{Et\sqrt{3}}{a3} = \frac{1}{3}\sqrt{3}\frac{Et}{a}$$

which was to be proved.

Answer to question 1b

Shells are very sensitive to imperfections (eccentricities, initial stresses, shrinkage, creep, inhomogeneous) because the theoretical strength is much reduced after buckling.



Answer to question 1c

For the membrane state it holds that

$$\frac{N_x}{R_x} + \frac{N_y}{R_y} = p_z\,.$$

In our case this becomes $\frac{2N}{a} = p$, so $N = \frac{pa}{2}$

$$\sigma = \frac{N}{t} = \frac{pa}{2t} \qquad (*3)$$

The stress should be much smaller than the buckling stress.

$$\gamma_b \sigma = \sigma_{cr}$$

Substitution of (*2) and (*3) in this gives

$$\gamma_b \frac{pa}{2t} = \frac{1}{\sqrt{3}} \frac{Et}{a}$$

from which we solve

$$t^2 = \frac{\sqrt{3}}{2} \frac{\gamma_b p a^2}{E}.$$

The water pressure is $p = \rho g d$. Therefore,

$$t^2 = \frac{\sqrt{3}}{2} \frac{\gamma_b \rho g d a^2}{E}$$

which was to be proved.

Answer to question 1d

$$p = \rho g d = 1000 \times 10 \times 400 = 4 \times 10^6 \text{ N/m}^2 = 4 \text{ N/mm}^2$$

$$\sigma = \frac{pa}{2t} = \frac{4 \times 1500}{2t} = \frac{3000}{t} \text{ N/mm}^2 \left[t \text{ in mm} \right]$$

Strength

$$\sigma_{allowable} = \frac{\sigma_y}{\gamma_s} = \frac{355}{1.6} = 222 \text{ N/mm}^2$$

$$\sigma = \frac{3000}{t} = 222 \rightarrow t = 13.5 \text{ mm}$$

Buckling

$$\sigma_{allowable} = \frac{\sigma_{cr}}{\gamma_b} = \frac{1}{7\sqrt{3}} \frac{Et}{a} = \frac{1}{7\sqrt{3}} \frac{2.1 \times 10^5}{1500} t = 11.5t \text{ N/mm}^2 \left[t \text{ in mm} \right]$$

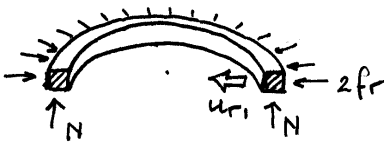
$$\sigma = \frac{3000}{t} = 11.5t \rightarrow t^2 = 260 \rightarrow t = 16.1 \text{ mm}$$

Conclusion: Buckling is determining. $t = 16.1$ mm

Answer to question 2a

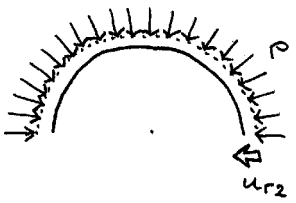
This problem can be divided in three parts, 1) deformation of the ring, 2) membrane deformation of the sphere and 3) bending deformation of the sphere. The deformation of the ring is

$$u_{r1} = \frac{2f_r a^2}{EA}. \qquad (*4)$$



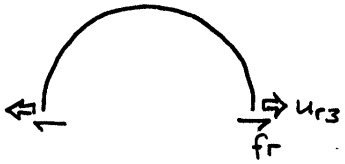
This can be derived from $N = 2f_r a$, $N = EA\varepsilon$ and $u_{r1} = a\varepsilon$. The membrane deformation is

$$u_{r2} = \frac{pa^2}{2Et}. \qquad (*5)$$



This can be derived from (9.2), (9.4) and (*3). The bending deformation is

$$u_{r3} = \frac{3f_r}{Et^3\mu^3} \qquad (*6)$$



and the bending moment is

$$t_x = \frac{f_r}{2\mu}. \qquad (*7)$$

These can be derived from (10.25) where $\varphi_x = 0$ due to symmetry. The gap between the membrane deformation and the ring deformation is closed by the bending deformation.

$$u_{r2} - u_{r1} = u_{r3}$$

Substitution of (*4), (*5) and (*6) in the latter gives

$$\frac{pa^2}{2Et} - \frac{2f_r a^2}{EA} = \frac{3f_r}{Et^3\mu^3}.$$

From this we solve f_r

$$\frac{pa^2}{2t} = f_r\left(\frac{3}{t^3\mu^3} + \frac{2a^2}{A}\right)$$

$$f_r = \frac{p}{\frac{6}{t^2a^2\mu^3} + \frac{4t}{A}}.$$

Substituting the latter in (*7) we obtain

$$t_x = \frac{f_r}{2\mu} = \frac{p}{\frac{12}{t^2a^2\mu^2} + \frac{8t\mu}{A}}.$$

Finally, the bending stress is

$$\sigma_{xx} = \frac{t_x}{\frac{1}{6}t^2} = \frac{p}{\frac{2}{a^2\mu^2} + \frac{4t^3\mu}{3A}} = \frac{\frac{p}{\mu^2}}{\frac{2}{a^2\mu^4} + \frac{4t^3}{3A\mu}} = \frac{\frac{pat}{\sqrt{3}}}{\frac{2a^2t^2}{a^2 3} + \frac{4t^3}{3A\mu}} = \frac{\sqrt{3}}{2 + \frac{4t}{A\mu}}\frac{a}{t}p$$

where it was used that

$$\mu^2 = \frac{\sqrt{3}}{at}.$$

Answer to question 2b

$$\sigma_{xx} = \frac{\sqrt{3}}{2 + \frac{4t}{\mu A}}\frac{a}{t}p$$

$$A = 4\frac{t}{\mu} \quad \Rightarrow \quad \frac{4t}{\mu A} = 1$$

$$\sigma_{bending} = \pm\frac{\sqrt{3}}{2+1}\frac{1500}{20}4 = \pm 173 \text{ N/mm}^2$$

$$\sigma_{membrane} = -\frac{pa}{2t} = -\frac{4\times 1500}{2\times 20} = -150 \text{ N/mm}^2$$

$$\sigma^+ = +173 - 150 = 23 \text{ N/mm}^2$$

$$\sigma^- = -173 - 150 = -323 \text{ N/mm}^2$$

Answer to question 3a

Boundary conditions at the straight edges

$$y_m = -\theta_o a, \theta_o a \quad \rightarrow \quad n_{yy} = n_{xy} = m_{yy} = v_y + \frac{\partial m_{xy}}{\partial x} = 0$$

Boundary conditions at the curved edges (Not an exam question)

$$x_m = -\tfrac{1}{2}l, \tfrac{1}{2}l \quad \rightarrow \quad u_z = u_y = n_{xx} = m_{xx} = 0$$

Answer to question 3b

Stress at the middle of the straight edge (p. 130)

$x = 0, y_b = 0$

$p = 3.5 \text{ kN/m}^2$

$l = 40 \text{ m}$

$\theta_o = 30^o \quad \cos 30^o = 0.866 \quad \sin 30^o = 0.500$

$t = 0.08 \text{ m}$

$a = 16 \text{ m}$

$$\gamma = \frac{\pi}{\sqrt[4]{3}}\frac{a}{l}\sqrt{\frac{t}{a}} = \frac{\pi}{\sqrt[4]{3}}\frac{16}{40}\sqrt{\frac{0.08}{16}} = 0.0675$$

$$\frac{\beta}{\alpha_n} = \sqrt[4]{\frac{t}{a}} = \sqrt[4]{\frac{0.08}{16}} = 0.266$$

Bending

$$\sigma_{xx,b(y_b=0)} = \frac{1}{\pi^2} p \frac{l}{t} \cos\frac{\pi x}{l}\left[4\sqrt[4]{3}\frac{11.31+16.98\,\gamma}{2.34+8.49\,\gamma}\sqrt{\frac{a}{t}}\cos\theta_o - 8\frac{7.27+18.64\,\gamma}{2.34+8.49\,\gamma}\frac{\beta}{\alpha_n}\sin\theta_o \right]$$

$$= \frac{1}{\pi^2} 3500 \frac{40}{0.08}\left[4\sqrt[4]{3}\frac{11.31+16.98\times 0.0675}{2.34+8.49\times 0.0675}\sqrt{\frac{16}{0.08}}0.866 - 8\frac{7.27+18.64\times 0.0675}{2.34+8.49\times 0.0675}0.266\times 0.500 \right]$$

$$= 39.33\,10^6 \text{ N/m}^2 = 39.33 \text{ N/mm}^2$$

Membrane

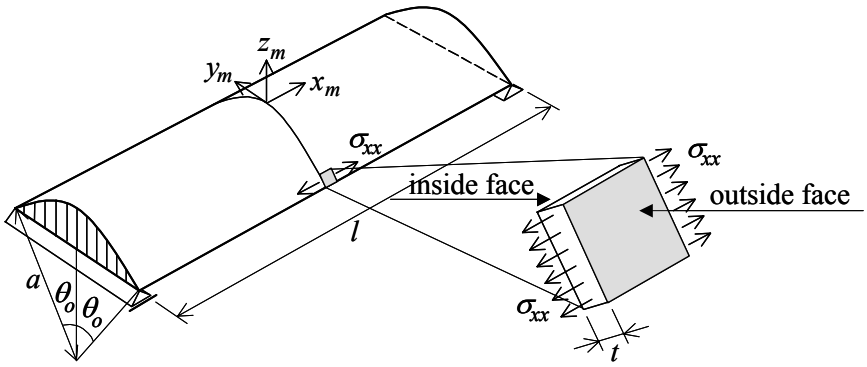
$$\sigma_{xx,m(y_b=0)} = \frac{1}{\pi^2} p \frac{l}{t} \cos\frac{\pi x}{l}\left[-\frac{8}{\pi}\frac{l}{a}\cos\theta_o \right]$$

$$= \frac{1}{\pi^2} 3500 \frac{40}{0.08}\left[-\frac{8}{\pi}\frac{40}{16}0.866 \right]$$

$$= -975\,000 \text{ N/m}^2 = -0.98 \text{ N/mm}^2$$

Total

$$\sigma_{xx(y_b=0)} = 39.33 - 0.98 = 38.35 \text{ N/mm}^2$$



Answer to question 3c

The tensile stress of 38.4 N/mm^2 must be carried by reinforcing steel with a yield strength of 435 N/mm^2. This means that the percentage of reinforcement has to be 38.4 / 435 × 100 = 8.8 %. This is an unrealistic high amount.

Increasing the thickness causes an increase of the stresses. This will not be a solution.
Possible solutions are:

- Increasing the radius
- Application of an edge beam
- Application of post-tensioning.