Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT5143 Stresses in Shells
Monday 23 January 2006, 14:00 – 17:00 hours

Make a selection of the following problems such that the summation of points is 10.
It is not allowed to solve more than these selected problems.

Problem 1 (2 points, 30 minutes)

Explain shortly the meaning of the following ten words in relation to structural shells.
Anticlastic, Membrane action, Meridian, Trajectory, Edge disturbance, Influence length, Boundary condition, Concentrated shear force, In-extensional deformation, Elpar.

Problem 2 (2 points, 30 minutes)

A shell is modelled in Rhinoceros. The straight edges have a length of 6 m. The curved edge has a radius of 3 m. The shell thickness is 70 mm. The Gaussian curvature is plotted in Figure 1. The minimum radius – related to the largest principal curvature – is plotted in Figure 2. Estimate the influence lengths of edge disturbances at each of the four edges.

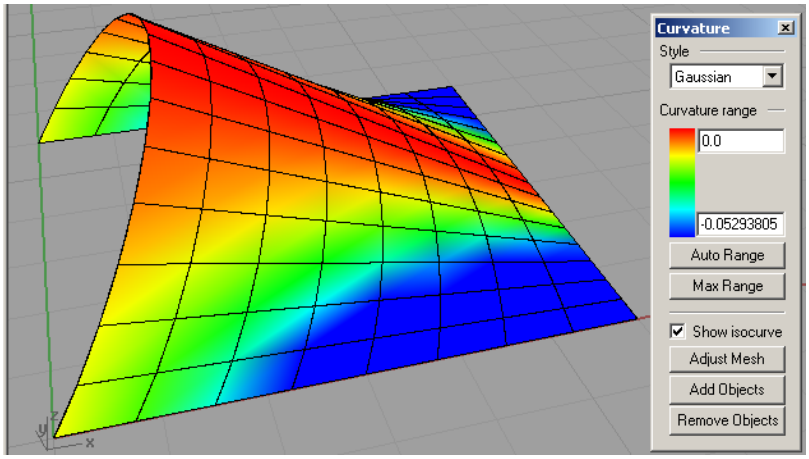


Figure 1. Gaussian curvature

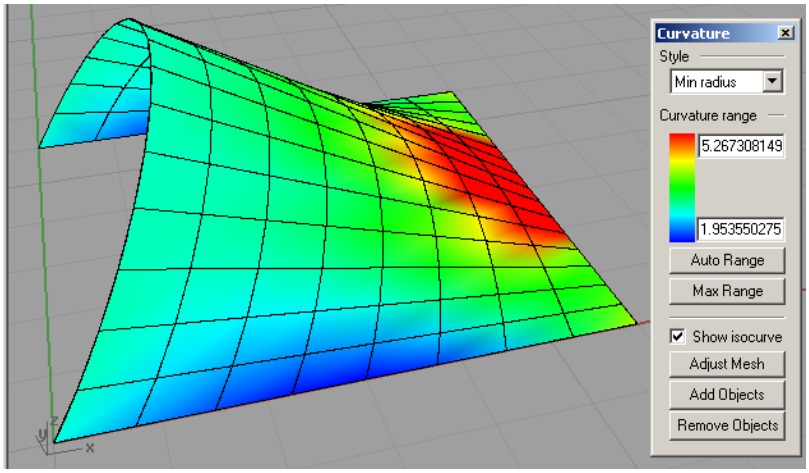


Figure 2. Minimum radius

Problem 3 (2 points, 30 minutes)

Derive a formula for the horizontal displacement *u* of the base of a spherical dome due to self-weight (Fig 3.).

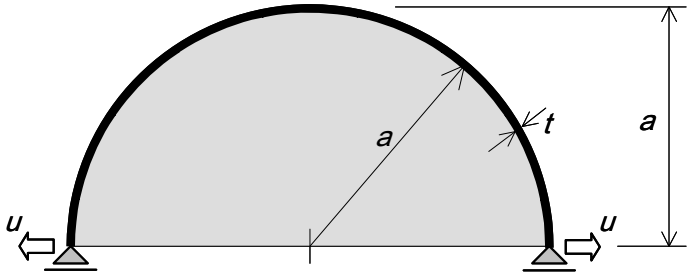


Figure 3. Dome loaded by self-weight

Problem 4 (2 points, 30 minutes)

Consider a designed shell in which self-weight is the dominant load. The designer increases the shell thickness everywhere by a factor two. Explain the following observations.

a The membrane stresses do not change.

b The largest bending stress changes a little (about 10%).

c The deflection does not change.

Problem 5 (2 points, 30 mininutes)

a How many support reactions (forces or moments) are needed at the least to fix a three-dimensional structure?

b What error does a structural analysis program give if less supports are used?

c Why do we need to be extra critical when using finite elements for shells?

d Which computation results do not depend on the coordinate system?
(Examples of coordinate systems are gobal *x*-*y*-*z*, local *x*-*y*-*z*, cylinder *r*-θ-*z*)

Problem 6 (2 points, 30 minutes)

Consider a shell that consists of four hypars (Fig. 4).

a Why does this shell need edge beams?

b Are the interior beams necessary? Explain your answer.

c Give the boundary conditions for one of the hypars.

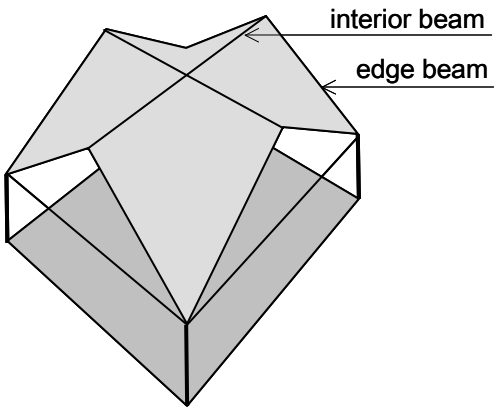


Figure 4. A four hypar shell

Problem 7 (4 points, 70 minutes)

A conical shell of reinforced concrete is supported by a ring-shaped edge beam of pre-stressed concrete (Fig. 5). The edge beam is supported by flexible columns. The shell thickness is t = 70 mm. The concrete has a Young's modulus $E = 3\ 10^7$ kN/m². The dead weight of the shell including roofing and finishing is p = 2.7 kN/m². The edge beam has a cross-section area A = 0.2 x 0.2 m² and is prestressed by a cable with a force P.

a Derive the following formulas for the distributed membrane forces in a cone.

$$n_{\phi\phi} = -\tfrac{1}{2} p\, x \frac{1}{\sin\phi}, \qquad n_{\theta\theta} = -p\, x \frac{\cos^2\phi}{\sin\phi}.$$

(In the derivation you can use any valid equation from the lecture book.)

b Calculate the membrane stresses in the shell next to the edge beam.

c The prestress force P is applied such that an edge disturbance does not occur. Derive the following formula for this prestress force

$$P = -\sigma_{\theta\theta}\left(\tfrac{1}{2} l\, t + A\right)$$

where $\sigma_{\theta\theta}$ is the hoop stress at the base of the cone and l the distance from the top to the base of the cone. Assume in the derivation that Poisson's ratio is zero, ν = 0.

d Calculate the prestress force P.

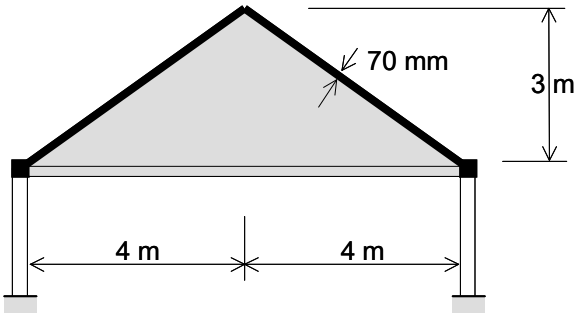


Figure 5. Conical shell structure

Problem 8 (4 points, 70 mininutes)

A double hypar shell is supported by three columns (Fig. 6). The shell thickness is t = 100 mm.The dead weight of the shell including roofing and finishing is p = 3.2 kN/m².

a Calculate the membrane forces and largest bending moment in the shell.

b Calculate the forces in the edge beams, cables and columns.

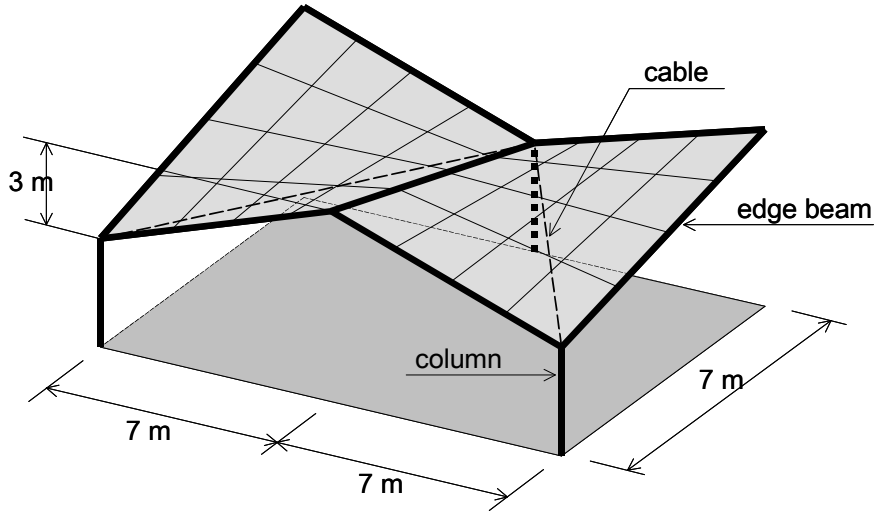


Figure 6. Double hypar shell

Exam CT5143, 23 January 2006
Solution of Problem 1

Anticlastic …….	The property of having everywhere a negative Gaussian curvature, for example a hypar
Membrane action ……..	A way of carrying load by n_{xx}, n_{yy} and n_{xy} only
Meridian …….	A line from the top to the base of a shell of revolution
Trajectory ……...	A line that connects principal directions
Edge disturbance ……..	Large stresses near a shell edge in comparison to some distance from the edge
Influence length …….	The distance from a disturbance at which the effects of the disturbance can be neglected
Boundary condition ……..	Mathematical formulation of a physical quantity at an edge for solving a differential equation
Concentrated shear force …	The interpretation of m_{xy} at the edges of a shell
In-extensional deformation ..	A way of deforming without activating membrane action
Elpar …….	Elliptical paraboloid, obtained by translating a parabola along another parabola

Solution of Problem 2

Looking from the curved edge the shell resembles a cylinder with a radius of 3 m.

$$l_i = 2.4\sqrt{at} = 2.4\sqrt{3 \times 0.07} = 1.1\text{m}.$$

Looking from the middle of the right-hand edge the shell resembles a cylinder with a radius of 5.26 m.

$$l_i = 2.4\sqrt{at} = 2.4\sqrt{5.26 \times 0.07} = 1.5\,\text{m}.$$

Looking from the front edge and back edge the shell resembles a hypar. This is confirmed by the negative Gausian curvature. The edge length is 6 m and the local minimum radius is 1.95 m

$$l_i = 2.25\sqrt[3]{atl} = 2.25\sqrt[3]{1.95 \times 0.07 \times 6} = 2.1\text{m}.$$

Solution of Problem 3

At page 147 of the lecture book we find the membrane forces due to self-weight

$$n_{\phi\phi} = \frac{-pa}{1+\cos\phi}$$

$$n_{\theta\theta} = pa\left(\frac{1}{1+\cos\phi} - \cos\phi\right)$$

Equation 9.9 is used to calculate the strain

$$\varepsilon_{\theta\theta} = \frac{1}{Et}\left(n_{\theta\theta} - \nu n_{\phi\phi}\right) = \frac{1}{Et}\left(pa\left(\frac{1}{1+\cos\phi} - \cos\phi\right) - \nu\frac{-pa}{1+\cos\phi}\right) = \frac{pa}{Et}\left(\frac{1+\nu}{1+\cos\phi} - \cos\phi\right)$$

The displacement is found using equation 9.11

$$u_r = \varepsilon_{\theta\theta} r = \frac{pa}{Et}\left(\frac{1+\nu}{1+\cos\frac{1}{2}\pi} - \cos\frac{1}{2}\pi\right)a = \frac{pa^2}{Et}(1+\nu)$$

Solution of Problem 4

a When the thickness is twice as large then the loading is twice as large and the membrane distributed forces are twice as large (linear elastic). The membrane stress remains the same because both distributed force as thickness are twice as large.

b The moments are often due to imposed deformation. For example in a hypar shell the edge moment is $m_{xx} = 0.511\frac{p}{l^2}\sqrt[3]{(atl)^4}$ The bending stress is $\sigma_{xx} = \frac{m_{xx}}{\frac{1}{6}t^2}$. The loading is $p = \rho g t$, where ρ is the mass density and g is the gravitation acceleration. Therefore, $\sigma_{xx} \sim \frac{t\sqrt[3]{t^4}}{t^2}$ (" $\sim$ " is "proportional to").

$$\frac{\sigma_{xx2}}{\sigma_{xx1}} = \frac{\frac{2t\sqrt[3]{(2t)^4}}{(2t)^2}}{\frac{t\sqrt[3]{t^4}}{t^2}} = \frac{2\sqrt[3]{2^4}}{2^2} = 1.26\ .$$

So, the bending stress increases when the shell thickness is doubled, however, only by 26%.

c The global deformation is determined mostly by the membrane action. Since the membrane stresses do not change the deflection does not change.

Solution of Problem 5

a Six forces or moments.

b The program will reply that there is a problem in inverting the stiffness matrix.

c The inner working of any shell element is very complex. In some situations an element will lock (high stiffness) or show unrealistic stresses.

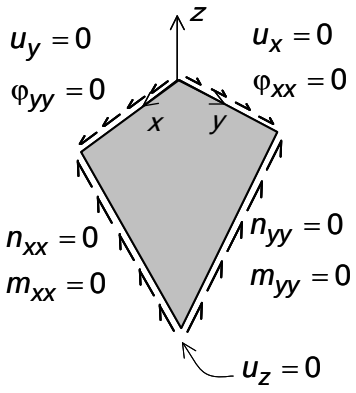
d Independent of the coordinate system are displacements, principal values of normal forces and moments, resulting shear force, Von Misses stress.

Solution of Problem 6

a A hypar shell carries the load by distributed shear forces nxy. The edge beams carry these forces from the shell edges to the columns.

b The interior beams are needed to carry the shear forces at the top edges of the hypar shells.

c



Solution of Problem 7

a Cone equations

For any shell of revolution the distributed membrane forces are (Eq. 9.7)

$$n_{\phi\phi} = \frac{1}{r \sin\phi} F\,, \qquad n_{\theta\theta} = p_z r_2 - \frac{1}{\sin^2\phi}\frac{F}{r_1}\,, \qquad F = \int (p_z \cos\phi - p_\phi \sin\phi) r\, r_1 d\phi\,.$$

The self-weight p is decomposed in the z and ϕ direction,.

$$p_\phi = p \sin\phi\,, \quad p_z = -p\cos\phi\,.$$

For a cone

$$r = x\cos\phi\,, \qquad r_1 = \infty\,. \qquad r_2 = \frac{x}{\tan\phi}\,, \qquad r_1 d\phi = dx\,.$$

Substitution in Eq. 9.7 gives

$$n_{\phi\phi} = \frac{1}{x\cos\phi\sin\phi} F\,, \qquad n_{\theta\theta} = -p\cos\phi\frac{x}{\tan\phi}\,, \qquad F = \int -p\,x\cos\phi\, dx = -p\,\tfrac{1}{2}x^2\cos\phi + C\,.$$

If $x = 0$ than $n_{\phi\phi} = 0$ so C = 0.

Finally, we obtain

$$n_{\phi\phi} = -\tfrac{1}{2} p\,x\frac{1}{\sin\phi}\,, \qquad n_{\theta\theta} = -p\,x\frac{\cos^2\phi}{\sin\phi} \qquad \text{Q.E.D.}$$

b Membrane stresses

$$l = \sqrt{4^2 + 3^2} = 5\,\text{m} \qquad \cos\phi = \frac{4}{5} \qquad \sin\phi = \frac{3}{5}$$

$$n_{\phi\phi} = -\tfrac{1}{2} p l \frac{1}{\sin\phi} = -\tfrac{1}{2} 2.7 \times 5 \frac{1}{\frac{3}{5}} = -11.25 \text{ kN/m}$$

$$n_{\theta\theta} = -p l \frac{\cos^2\phi}{\sin\phi} = -2.7 \times 5 \frac{\frac{16}{25}}{\frac{3}{5}} = -14.40 \text{ kN/m}$$

$$\sigma_{\phi\phi} = \frac{n_{\phi\phi}}{t} = \frac{-11.25}{0.07} = -161 \text{ kN/m}^2 = -0.161 \text{ N/mm}^2$$

$$\sigma_{\theta\theta} = \frac{n_{\theta\theta}}{t} = \frac{-14.40}{0.07} = -205 \text{ kN/m}^2 = -0.205 \text{ N/mm}^2$$

c Prestress formula

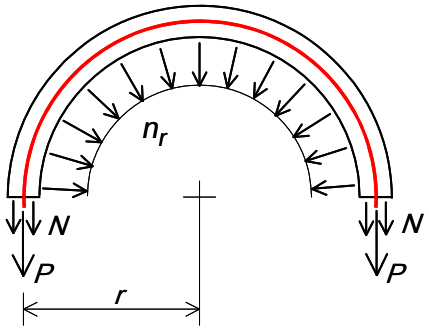
The horizontal load on the cone edge is $n_r = n_{\phi\phi}\cos\phi$. This gives a force in the edge beam of $N + P = -n_r\, r$ where $r = l\cos\phi$. Therefore,

$N + P = -n_{\phi\phi}\, l \cos^2\phi$.

In order for the edge beam to have the same deformation as the cone edge the stresses need to be the same (ν is neglected here).

$$\frac{N}{A} = \sigma_{\theta\theta}$$

Therefore,



$$2P + 2N + n_r\, 2r = 0$$

$$P = -n_{\phi\phi}\, l \cos^2\phi - \sigma_{\theta\theta} A = \tfrac{1}{2} p l \frac{1}{\sin\phi} l \cos^2\phi - \sigma_{\theta\theta} A =$$

$$= -\tfrac{1}{2} n_{\theta\theta}\, l - \sigma_{\theta\theta} A = -\tfrac{1}{2}\sigma_{\theta\theta}\, l\, t - \sigma_{\theta\theta} A = -\sigma_{\theta\theta}\left(\tfrac{1}{2} l\, t + A\right)$$

Q.E.D.

d Prestress force

$$P = -\sigma_{\theta\theta}\left(\tfrac{1}{2} l\, t + A\right) = 0.205\left(\tfrac{1}{2} 5000 \times 70 + 200 \times 200\right) = 44075 \text{ N.}$$

Solution of Problem 8

a Stresses

The radius a of both hypars is $z = \frac{xy}{a} \Rightarrow 1.5 = \frac{3.5 \times 3.5}{a} \Rightarrow a = \frac{3.5 \times 3.5}{1.5} = 8.17$ m.

The direction of the dead load is approximately perpendicular to the shell surface. Therefore the distributed membrane force is $n_{xy} = \frac{1}{2} a\, p_z = \frac{1}{2} 8.17 \times 3.2 = 13.07$ kN/m.

The largest moment in the shell occurs at the interior beam because there rotation will not occur due to symmetry. The moment is

$$m_{xx} = 0.511 \frac{p}{l^2} \sqrt[3]{(a\, t\, l)^4} = 0.511 \frac{3.2}{7.62^2} \sqrt[3]{(8.17 \times 0.1 \times 7.62)^4} = 0.323 \text{ kNm/m.}$$

b Force flow

The length of the edge beams is $l = \sqrt{7^2 + 3^2} = 7.62$ m.

The normal force in the bottom of the edge beams is $N_e = 7.62 \times 13.07 = 99.59$ kN.

The force in the corner columns is $R_e = 2 \frac{3}{7.62} 99.59 = 78.42$ kN.

The force in the cables is $F = \sqrt{2} \frac{7}{7.62} 99.59 = 129.38$ kN.

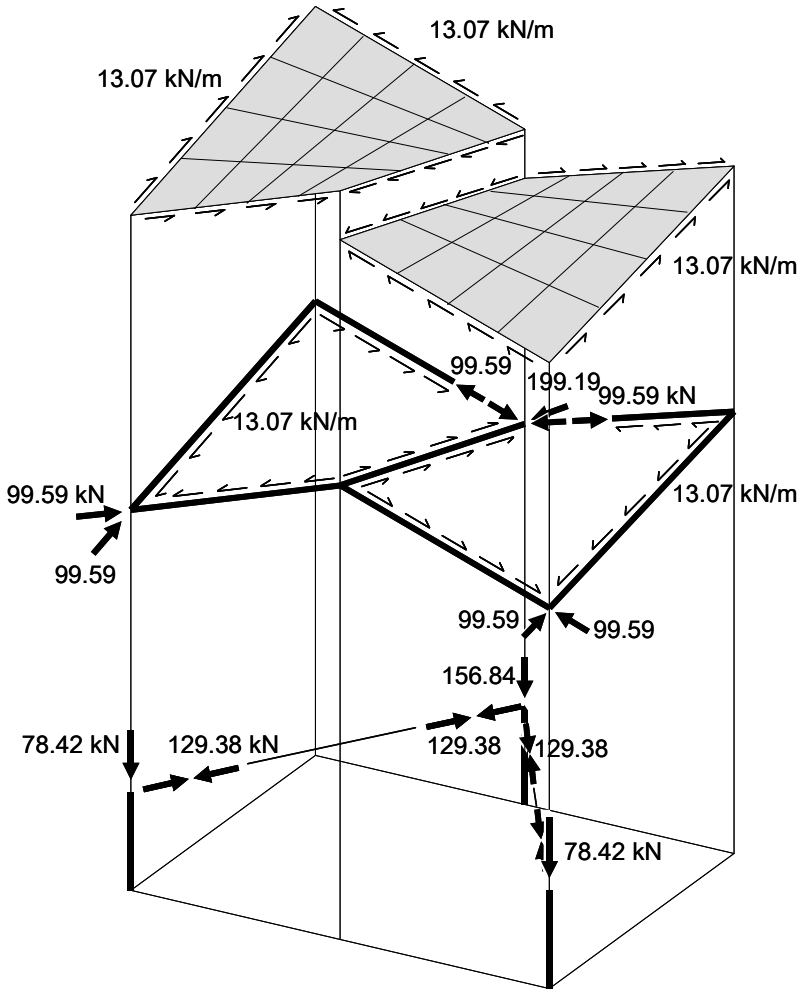
The normal force in the bottom of the interior beam is $N_i = 2 \times 7.62 \times 13.07 = 199.19$ kN.

The normal force in the edge column is

$$R_i = \frac{3}{7.62} \times 199.19 + \frac{3}{7.62} 99.59 + \frac{3}{7.62} 99.59 = 156.84 \text{ kN.}$$

As a check we calculate the total load $G = 2 \times 7 \times 7 \times 3.2 = 314$ kN.

The total support reaction is $G = 2 \times 78.42 + 154.84 = 311.68$ kN which is approximately equal to the load.


13.07 kN/m
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99.59
199.19
99.59 kN
13.07 kN/m
13.07 kN/m
99.59 kN
99.59
99.59
99.59
156.84
78.42 kN
129.38 kN
129.38
129.38
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