Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT5143 Stresses in Shells
Monday 3 July 2006, 14:00 – 17:00 hours

Problem 1 (2 points, 35 minutes)

Explain shortly the meaning of the following eight words in relation to structural shells.

Synclastic
Bending action
Hoop direction
Compatibility condition
Hypar
Kinematic equations
Homogeneous solution
Gaussian curvature

Problem 2 (2 points, 35 minutes)

a Figure 1 shows a representative shell part. Draw the moments and forces on the visible sections in the positive directions.

b The shell differential equation is derived in a local reference system *x-y-z* that follows the shape of the shell. However, the shell shape is defined in a global reference system, confusingly also referred to as *x-y-z*. Why does not this give problems in the shells analysed during the course?

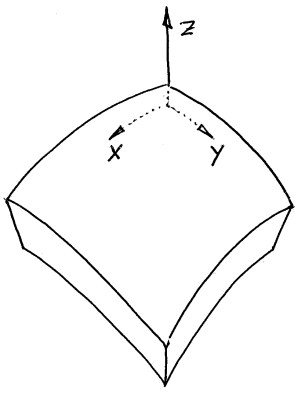


Figure 1. Representative shell part

Problem 3 (2 points, 35 minutes)

We consider a toped cone loaded by self-weight (Fig. 2).

a Show that the force flow in the cone is described by

$$n_{\phi\phi} = -p\frac{r^2 - c^2}{2r\sin\phi\cos\phi}$$

$$n_{\theta\theta} = -p\frac{r}{\tan\phi}$$

b Derive a formula for the horizontal displacement *u* of the base of this toped cone.

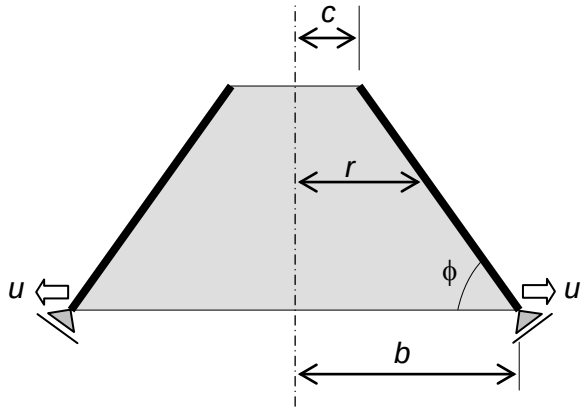


Figure 2. Cross-section of a toped cone loaded by self-weight

Problem 4 (2 points, 35 minutes)

We compare the behaviour of an arch and a shell.

a Moments in arches can be necessary to carry part of the loading. However, in general moments in shells do not carry any loading. What causes moments in a shell?

b The optimal arch shape can be found by hanging a chain between two points. In this we define optimal as having no moments due to gravity loading. Similarly, an optimal shell shape can be found by hanging a towel between three or more points. However, other shell shapes that cannot be obtained with a towel are equally optimal. Why is this so?

Problem 5 (2 points, 35 mininutes)

An assembly of four identical hypar shells is supported by four columns (Fig. 3). The shell thickness is t = 100 mm. The dead weight of the shell including roofing and finishing is p = 3.5 kN/m².

a Calculate the membrane forces and largest bending moment in the shell.

b Calculate the forces in the edge beams, cables and columns.

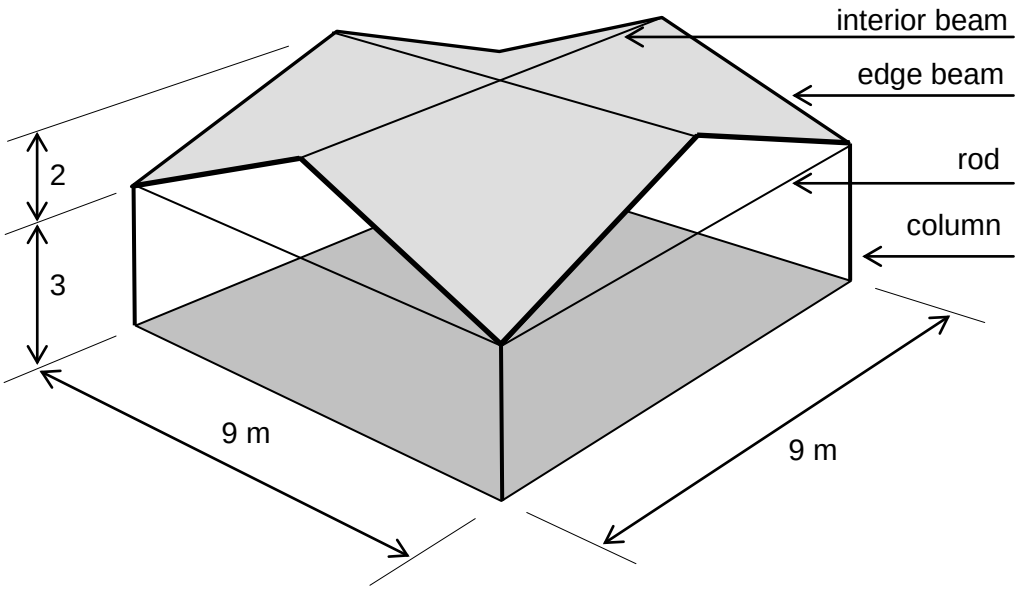


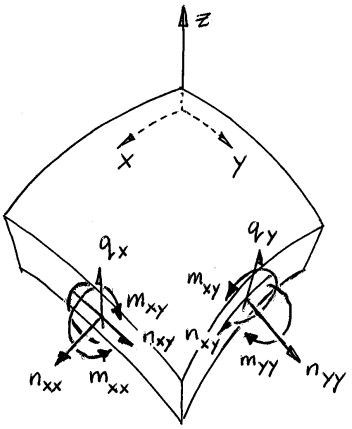
Figure 3. Four hypar shell

Exam CT5143, 23 January 2006
Solution of Problem 1

Synclastic …….	The property of having everywhere a positive Gaussian curvature, for example an elpar
Bending action …….	A way of carrying load by m_{xx}, m_{yy} and m_{xy} only
Hoop direction …….	Pointing parallel to the base of a shell of revolution and tangent to the surface.
Compatibility condition ……	Condition that the strains need to fulfil to prevent gaps and overlaps in the material
Hypar …….	Hyperbolical paraboloid, obtained by translating a parabola along another parabola
Kinematic equations …….	Mathematical relation between strains and displacements
Homogeneous solution ……	Solution of the differential equation for zero loading.
Gaussian curvature ……..	Product of the principal curvatures $k_G = k_1 k_2$

Solution of Problem 2

a Section forces and moments.



b We only need to know the shell curvatures k_x, k_y and k_{xy} in the local reference system.

The shells analysed in the course have an elementary shape for which these curvatures are obvious. Therefore we do not use the mathematical formulation of the shell shape.

Solution of Problem 3

The membrane solution of a shell of revolution is

$$F = \int \left(p_z \cos\phi - p_\phi \sin\phi \right) r \, r_1 d\phi$$

$$n_{\phi\phi} = \frac{1}{r \sin\phi} F \qquad \text{(Eqs 9.7)}$$

$$n_{\theta\theta} = p_z r_2 - \frac{1}{\sin^2\phi} \frac{F}{r_1}$$

Gravity loading has the following components

$$p_\phi = p\sin\phi$$
$$p_z = -p\cos\phi \qquad \text{(p. 146)}$$

The shape of a cone has the following properties

$$r_1 = \infty$$
$$r_1 d\phi = dx \qquad \text{(p. 145)}$$
$$r_2 = \frac{r}{\sin\phi}$$

Substitution in the membrane solution gives.

$$F = \int\left(-p\cos^2\phi - p\sin^2\phi\right) r\,dx = -p\int r\,dx = -p\int_{x=0}^{\frac{r-c}{\cos\phi}} (c + x\cos\phi)\,dx = -p\left(cx + \tfrac{1}{2}x^2\cos\phi\right)_{x=0}^{\frac{r-c}{\cos\phi}}$$
$$= -p\left(c\frac{r-c}{\cos\phi} + \tfrac{1}{2}\left(\frac{r-c}{\cos\phi}\right)^2\cos\phi\right) = -p\left(\frac{cr-c^2}{\cos\phi} + \tfrac{1}{2}\frac{r^2-2rc+c^2}{\cos\phi}\right) = -p\frac{cr-c^2+\tfrac{1}{2}r^2-rc+\tfrac{1}{2}c^2}{\cos\phi}$$
$$= -p\frac{r^2-c^2}{2\cos\phi}$$
$$n_{\phi\phi} = -p\frac{r^2-c^2}{2r\sin\phi\cos\phi}$$
$$n_{\theta\theta} = -p\cos\phi\frac{r}{\sin\phi} = -p\frac{r}{\tan\phi}$$

Q.E.D.

b At the base of the cone $r = b$.

$$n_{\phi\phi} = -p\frac{b^2-c^2}{2b\sin\phi\cos\phi}$$
$$n_{\theta\theta} = -p\frac{b}{\tan\phi}$$

Equation 9.9 is used to calculate the strain at the base

$$\varepsilon_{\theta\theta} = \frac{1}{Et}\left(n_{\theta\theta} - \nu n_{\phi\phi}\right) = \frac{1}{Et}\left(-p\frac{b}{\tan\phi} + \nu p\frac{b^2-c^2}{2b\sin\phi\cos\phi}\right) = \frac{-p}{Et}\left(\frac{b}{\tan\phi} - \nu\frac{b^2-c^2}{2b\sin\phi\cos\phi}\right)$$

The displacement is found using equation 9.11

$$u_r = \varepsilon_{\theta\theta} r = \frac{-p}{Et}\left(\frac{b}{\tan\phi} - \nu\frac{b^2-c^2}{2b\sin\phi\cos\phi}\right) b = \frac{-p}{Et}\left(\frac{b^2}{\tan\phi} - \nu\frac{b^2-c^2}{2\sin\phi\cos\phi}\right)$$

Solution of Problem 4

a Everything still needs to fit after applying the load. For example, just the normal forces would create a displacement, however, the boundary conditions do not allow it. Closing the gap causes moments also referred to as compatibility moments.

b Provided that in-extensional deformations are not possible and the thickness is sufficient, any shell shape can carry any loading with just membrane action. So, it this respect all shells are optimal.

Solution of Problem 5

a Stresses

The radius *a* of a hypar is $z = \frac{x\,y}{a} \Rightarrow 2 = \frac{4.5 \times 4.5}{a} \Rightarrow a = \frac{4.5 \times 4.5}{2} = 10.13$ m.

The direction of the dead load is approximately perpendicular to the shell surface. Therefore the distributed membrane force is $n_{xy} = \frac{1}{2} a\, p_z = \frac{1}{2} 10.13 \times 3.5 = 17.73$ kN/m.

The largest moment in the shell occurs at the interior beam because there rotation will not occur due to symmetry. The moment is

$$m_{xx} = 0.511 \frac{p}{l^2} \sqrt[3]{(a\,t\,l)^4} = 0.511 \frac{3.5}{4.5^2} \sqrt[3]{(10.13 \times 0.1 \times 4.5)^4} = 0.668 \text{ kNm/m.}$$

b Force flow

The length of the edge beams is $l = \sqrt{4.5^2 + 2^2} = 4.92$ m.

The normal force in the bottom of the edge beams is $N_e = 4.92 \times 17.73 = 87.23$ kN.

The force in the columns is $R = 2 \frac{2}{4.92} 87.23 = 70.92$ kN.

The force in the cables is $F = \frac{4.5}{4.92} 87.23 = 79.78$ kN.

The normal force in the center of the interior beam is $N_i = 2 \times 17.73 \times 4.5 = 159.57$ kN.

As a check we calculate the total load $G = 9 \times 9 \times 3.5 = 283.50$ kN.
The total support reaction is $G = 4 \times 70.92 = 283.90$ kN which is approximately equal to the load.