Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT5143 Stresses in Shells
Monday 2 July 2007, 14:00 – 17:00 hours

Problem 1 (2 point, 35 minutes)

Explain shortly the meaning of the following eight words in relation to structural shells.

Mean curvature
Sagitta
Folded plate roof
Integration point
In-extensional deformation
Yoshimura pattern
Flexural rigidity
Influence length

Problem 2 (2 points, 35 minutes)

a Figure 1 shows a representative shell part with forces and moments in the positive directions. Add to this figure the variables for the directions, forces and moments (for example x, n_{xx} , ... Make clear which variable belongs to which arrow.)

b In a shell there is membrane action and bending action. By which specific equation are these two actions coupled?

c What is the difference between the shell curvature and the shell deformation curvature.

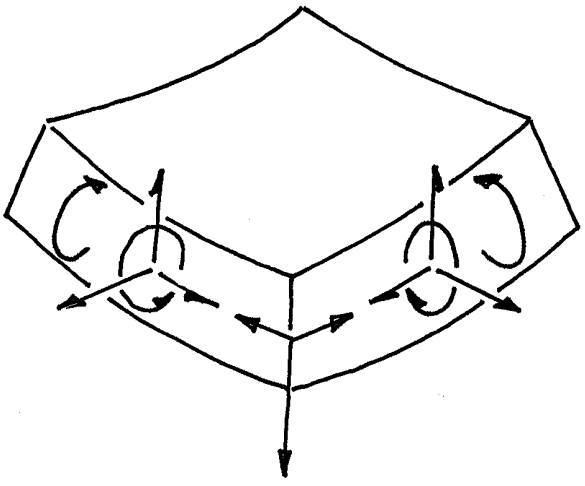


Figure 1. Representative shell part

d Gaussian curvature can be calculated by $k_G = k_x k_y - k_{xy}^2$. Show that the Gaussian curvature after deformations is $k_G = (k_x - \kappa_{xx})(k_y - \kappa_{yy}) - (k_{xy} - \frac{1}{2}\rho_{xy})^2$.

Problem 3 (2 points, 35 minutes)

We consider a conoid reinforced concrete shell loaded by self-weight (Fig. 2, 3, 4). The three straight edges are fixed. The curved edge is free. The shell is 60 m long and 60 m wide and has a height of 18 m. The thickness is 70 mm. The finite elements have a width of approximately 3 m.

a Are the finite elements sufficiently small? Explain your answer.

b What causes the moments in the shell?

c What will change in the results when cracking of the concrete is included in the analysis.

d How can this shell be improved?

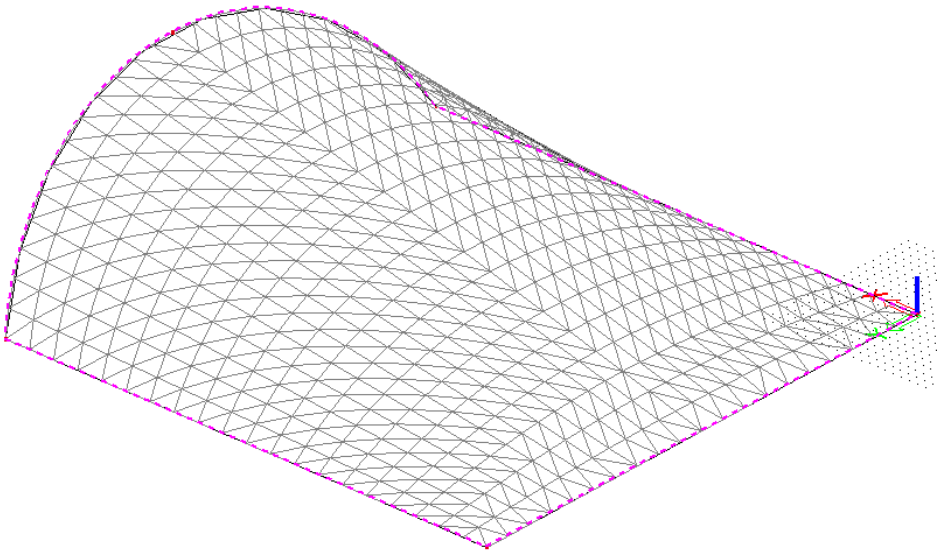


Figure 2. Conoid and finite element mesh

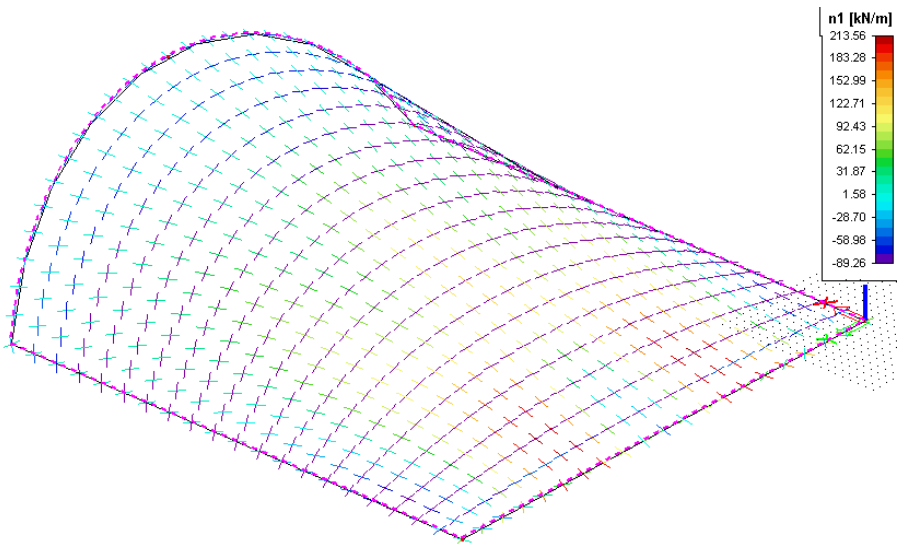


Figure 3. Principal directions of the membrane forces due to self-weight

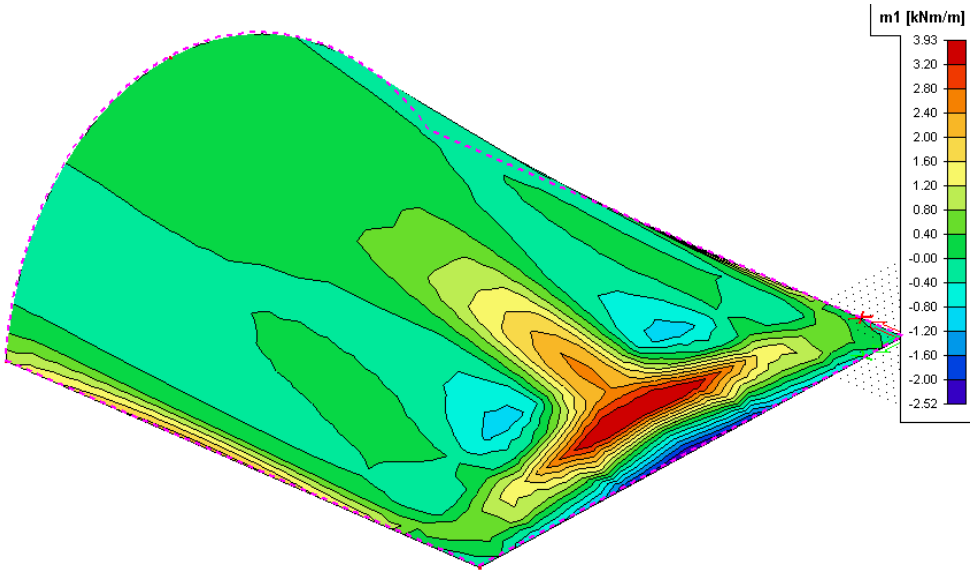


Figure 4. Largest principal value of the bending moment due to self-weight

Problem 4 (2 points, 35 minutes)

Check whether the cooling tower of Figure 5 will buckle due to self-weight.

Height 150 m
Bottom diameter 180 m
Wall thickness 0.10 m
Curvature in a vertical section 1/145 1/m
Mass 2400 kg/m³
Young's modulus 30000 N/mm² = 30000e6 N/m²
Gravity acceleration 10 m/s²
Knock down factor 1/6

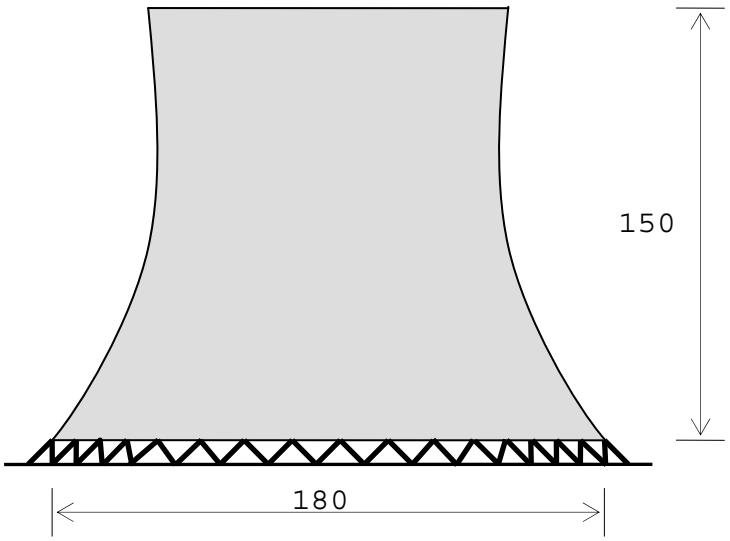


Figure 5. Cooling tower dimensions

Problem 5 (2 points, 35 mininutes)

An assembly of four identical hypar shells is supported by one column in the middle (Fig. 6). The shell thickness is t = 70 mm. The dead weight of the shell including roofing and finishing is p = 3.2 kN/m².

a Calculate the membrane forces and largest bending moment in the shell.

b Calculate the forces in the edge beams, interior beams and column.

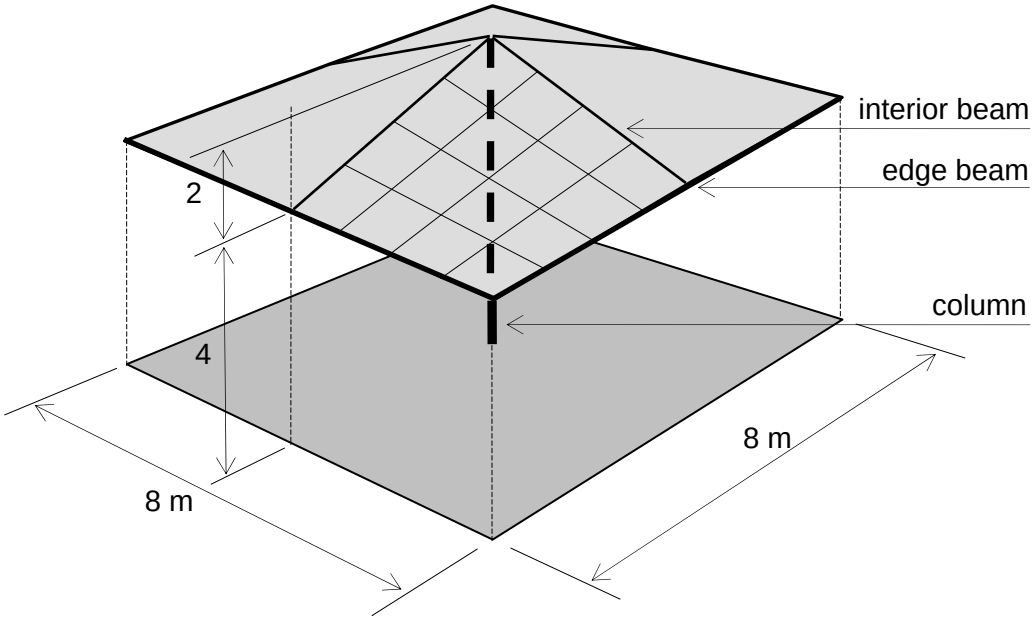


Figure 6. Four hypar shell

Exam CT5143, 2 July 2007
Solution of Problem 1

Mean curvature …….. $\frac{1}{2}(k_1 + k_2)$

Sagitta …….. Height of an arch

Folded plate roof …… Roof made of flat plates that are attached at angles.

Integration point …….. Point in a finite element where the material behaviour is computed.

In-extensional deformation … Deformation with no extension and only bending.

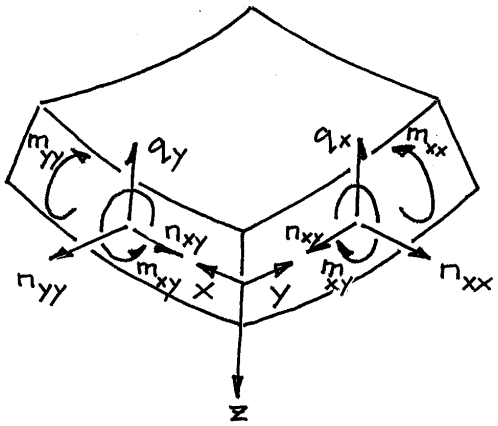
Yoshimura pattern …….. Plastic buckling shape of an axially compressed cylinder.

Flexural rigidity ..... $\frac{Et^3}{12(1-\nu^2)}$

Influence length …… The distance from a disturbance at which the influence of the disturbance can be neglected

Solution of Problem 2

a Variables



b The only coupling occurs in the equilibrium equation in the direction perpendicular to the surface.

c The shell curvatures are $k_x = \frac{\partial^2 z}{\partial x^2}, k_y = \frac{\partial^2 z}{\partial y^2}$ and $k_{xy} = \frac{\partial^2 z}{\partial x \partial y}$ in the local Cartesian coordinate system.

The shell deformation curvatures are $\kappa_{xx} = -\frac{\partial^2 u_z}{\partial x^2}, \kappa_{yy} = -\frac{\partial^2 u_z}{\partial y^2}$ and $\rho_{xy} = -2\frac{\partial^2 u_z}{\partial x \partial y}$ in the local (curved) coordinate system.

d The middle surface of a shell can be approximated locally by

$$z = \tfrac{1}{2}k_x x^2 + k_{xy} xy + \tfrac{1}{2}k_y y^2 .$$

The origin of the coordinate system is on the surface and the *z* axis is perpendicular to the surface. De deformation of a shell can be approximated locally by

$$u_z = u_{zo} + \varphi_y x + \varphi_x y - \tfrac{1}{2}\kappa_{xx} x^2 - \tfrac{1}{2}\rho_{xy} xy - \tfrac{1}{2}\kappa_{yy} y^2 .$$

After all, applying the above definitions of the deformation curvatures (problem 2c) produces the correct results. We obtain the deformed shape by

$$z_d = z + u_z$$
$$= u_{zo} + \varphi_y x + \varphi_x y + \tfrac{1}{2}(k_x - \kappa_{xx})x^2 + \tfrac{1}{2}(k_y - \kappa_{yy})y^2 + (k_{xy} - \tfrac{1}{2}\rho_{xy})xy$$

Apparently, the curvature of the deformed surface in the *x* direction is $k_x - \kappa_{xx}$. etc.
Therefore,

$$k_{Gd} = (k_x - \kappa_{xx})(k_y - \kappa_{yy}) - (k_{xy} - \tfrac{1}{2}\rho_{xy})^2 .$$

Q.E.D.

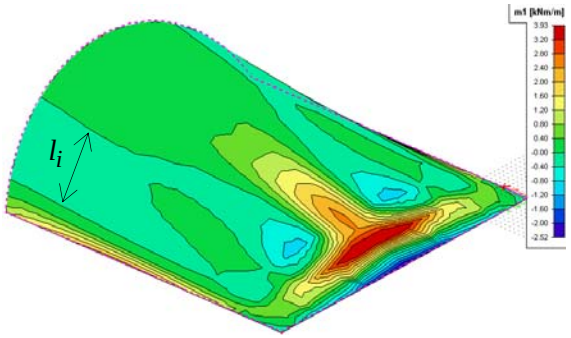
Encore (not an exam problem)
The increase of the Gaussian curvature is

$$\Gamma = k_{Gd} - k_G \approx -k_x\kappa_{yy} + k_{xy}\rho_{xy} - k_y\kappa_{xx} .$$

When this value is zero the local deformation is in-extensional. In-extensional deformation gives small stresses when a displacement is imposed (foundation settlement). In-extensional deformation gives large stresses when a force is imposed (snow load). Therefore, a contour plot of Γ shows where a shell design needs to be improved.

Solution of Problem 3

a The elements of 3 m are too large. From the moment figure we estimate that the influence length is 15 m (distance from 0 to 0). Over this length we need at least 6 elements, which would be 15/6 = 2.5 m wide.



(However, the best way to check the element size is repeat the analysis with half size elements. If the important results change little than the element size was OK.)

b The shell deforms due to the normal forces. However, at the straight edges it is pulled back to the foundation. This causes large deformation curvatures and therefore moments.
Also, at the lower part of the shell it is almost flat. It starts to act as a plate, which carries its load in bending.

c If it cracks the stiffness is reduced and the moments due to the fixed edges are reduced. The deflections and normal forces will be almost unchanged.

(Cracking is not only favourable because it also strongly reduces the bucking load of the shell.)

d If possible, change the fixed supports to hinges. Also give it more curvature in the almost flat part.

Solution of Problem 4

The largest stress occurs at the bottom of the shell. It can be estimated by assuming a cylinder shape.

stress = mass x g x height x thickness = 2400 x 10 x 150 x 0.1 = 360000 N/m = 360 kN/m

The critical normal force can be calculated by

$$n_{cr} = 0.6\frac{E t^2}{a} = 0.6\frac{30000 \times 0.1^2}{90} = 2 \times 10^6 \text{ N/m} = 2000 \text{ kN/m}$$

where, a is the vertical radius of the shell.

The shell is sensitive to imperfections and the real buckling force is found by multiplication with the knock down factor.

$$n_{buc} = \frac{1}{6}2000 = 330 \text{ kN/m}$$

The shell will buckle because the normal force (360 kN/m) is larger than the buckling normal force (330 kN/m).

Solution of Problem 5

a Stresses

The radius of the hypar is

$$a = \frac{b^2}{h} = \frac{4^2}{2} = 8 \text{ m.}$$

The direction of the dead load is approximately perpendicular to the shell surface. Therefore the distributed membrane force is

$$n_{xy} = \tfrac{1}{2} a p_z = \tfrac{1}{2} 8 \times 3.2 = 12.8 \text{ kN/m.}$$

The length of the interior beams is $l = \sqrt{4^2 + 2^2} = 4.47$ m.

The largest moment in the shell occurs at the interior beam because there rotation will not occur due to symmetry. The moment is

$$m_{xx} = 0.511\frac{p}{l^2}\sqrt[3]{(a t l)^4} = 0.511\frac{3.2}{4.47^2}\sqrt[3]{(8 \times 0.07 \times 4.47)^4} = 0.28 \text{ kNm/m.}$$

b Force flow

The normal force in the middle of the edge beams is

$N_e = 4 \times 12.8 = 51.2$ kN compression.

The normal force in the top of the interior beams is

$N_i = 2 \times 4.47 \times 12.8 = 114.4$ kN tension.

The force in the column is $R = 4\dfrac{2}{4.47}114.4 = 204.7$ kN.

As a check we calculate the total load $G = 8 \times 8 \times 3.2 = 204.8$ kN which is approximately equal to the column force.

