Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT4143 Shell Analysis
Exam CT5143 Shell Analysis
Monday 1 July 2010, 14:00 – 17:00 hours

If you take CT5143 instead of CT4143 then write this at the first of your answer sheets and skip problem 4 and 6.

Problem 1 (30 minutes)

Explain shortly the meaning of the following eight words in relation to structural shells.

Sagitta
Compatibility equation
Equivalent stress or Von Mises stress
CNIT
Principal directions
Minimal surface
Critical load
Yoshimura pattern

Problem 2 (30 minutes)

a Figure 1 shows a shell part with forces and moments in the positive directions. Add to this figure the variables for the directions, forces and moments (for example x, n_{xx} , ... Make clear which variable belongs to which arrow.)

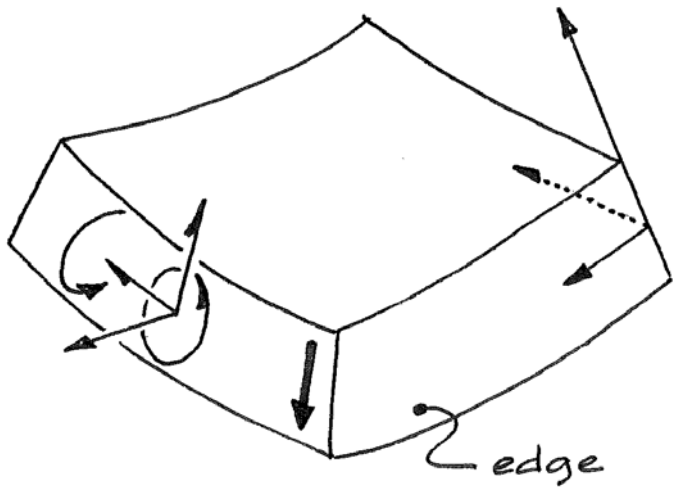


Figure 1. Representative shell part

b Which formula is used to calculate the shear stress σ_{yz} in the location $(x,y,z) = (0,0,0)$?

c Why do many shells have edge beams?

d What is a structural disadvantage of large edge beams?

e Suppose that a shell will be made of reinforced concrete and no edge beam will be applied. What reinforcement can be used to carry the concentrated shear force?

Problem 3 (30 minutes)

a The ultimate load of a thin reinforced concrete shell can depend strongly on cracking of the concrete. Why is this? Select one of the following answers.

A The cracks reduce the bending and membrane stiffness which are important for buckling.
B The cracks are extra imperfections of the shell geometry.
C The cracks reduce the tensile strength of the shell cross-section.
D The cracks allow water to reach the reinforcing bars which will corrode and loose strength.

b Many reinforced concrete shell structures have been build in the period 1955 to 1965. Why was this? Select one of the following answers.

A Labour was cheap.
B Architects liked the shapes.
C Engineers were trained in the mathematics that was required.
D A few leading engineers pushed it forward.
E People wanted to try new things.

c The Sanders-Koiter equations are not accurate for large deflections, for example at the top of a 50 m steel industrial chimney. However, finite element analyses of the chimney do not have this problem. Why is this? Select one of the following answers.

A Shell elements in most programs are not based on the Sanders-Koiter equations.
B Chimneys are analysed with volume elements not with shell elements.
C The deflection at the top is small compared to the length of the chimney.
D The elements are small compared to the influence length of the chimney edge.

d Suppose that the natural frequencies of a shell are too small. How can this be improved? Select one of the following answers.

A Increase the Gaussian curvature and increase the mass
B Decrease the Gaussian curvature and increase the mass
C Increase the Gaussian curvature and decrease the mass
D Decrease the Gaussian curvature and decrease the mass

e What is the influence length used for? Select one of the following answers.

A To calculate the factor β in the solution to the shell differential equation.
B To determine the finite element size near discontinuities.
C To calculate buckling lengths in the principal curvature directions.
D To predict imperfection sensitivity of very thin shell structures.

Problem 4 (30 minutes)

If the deformation of a shell is in-extensional then the Gaussian curvature does not change.[1]

a Prove this statement using the shell compatibility equation.

The reversed statement is not true. So, if the Gaussian curvature does not change this does not mean that the deformation is in-extensional.

b Give an example of the latter.

c Nonetheless, how can Gaussian curvature be used to spot in-extensional deformation?

Problem 5 (30 minutes)

We consider an assembly of four identical hypar shells (Fig. 4). The shell is supported at the four corners. These supports cannot carry horizontal forces. The design includes a free beam that is attached to the shell only at its start point and end point. The shell thickness is t = 40 mm. The dead weight of the shell including roofing and live load is 3 kN/m².

a Does this design need cables or tension rods? If so, add these to the design.

b The radius of curvature of the shell parts is a = 27.3 m. Calculate the membrane force and the largest bending moment in the shell.

c Calculate the forces in the edge beams, interior beams, free beam and cables. Also calculate the support reactions.

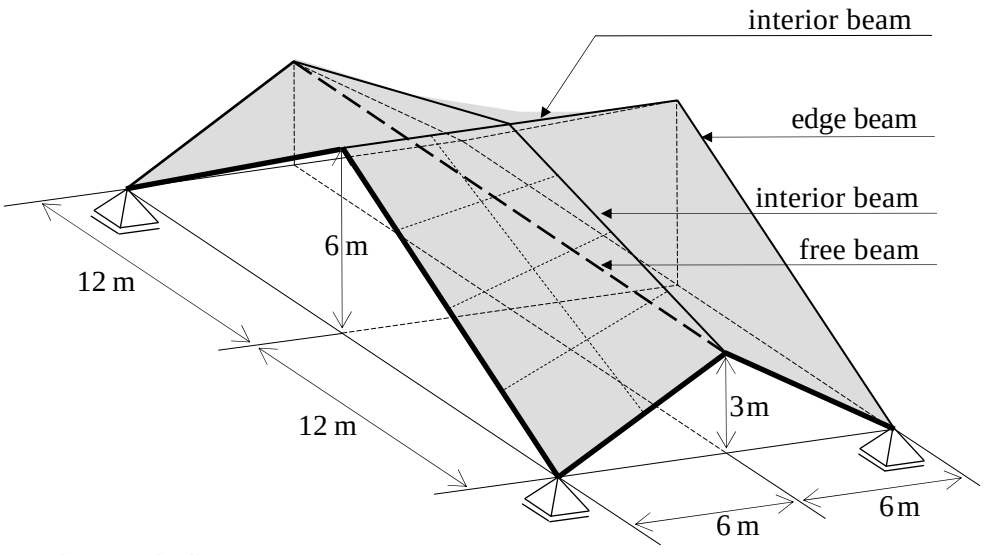


Figure 4. Four hypar shell

[1] This theorem was formulated and proved by J.C.J. Gauß (1777- 1855). It is called Theorema Egregium, which is Latin for "remarkable theorem". Prof. Gauß formulated it as "If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged." (translated from Latin)

d Check whether the shell will buckle. Use Young's modulus is 15 x 10^6 kN/m² (which is the tangent stiffness of cracked reinforced concrete at small stresses).

Problem 6 (30 minutes)

A roof structure consists of four hypar shells of 90 mm thickness (Fig. 5). The structure has edge beams and interior beams. It has been analysed for self weight. The principal directions of one hypar shell are shown in Fig. 6 and Fig.7. Draw the trajectories on these figures. Explain the force flow. Include the words "influence length" and "compatibility" in the explanation.

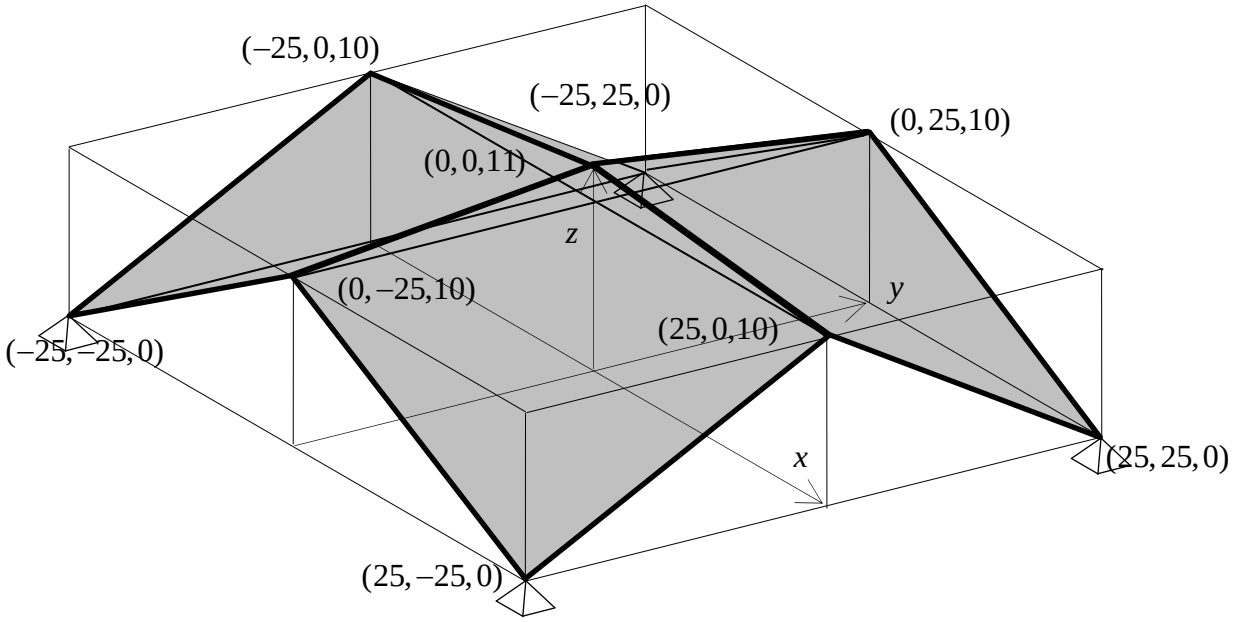


Figure 5. Structure consisting of four hypar shells

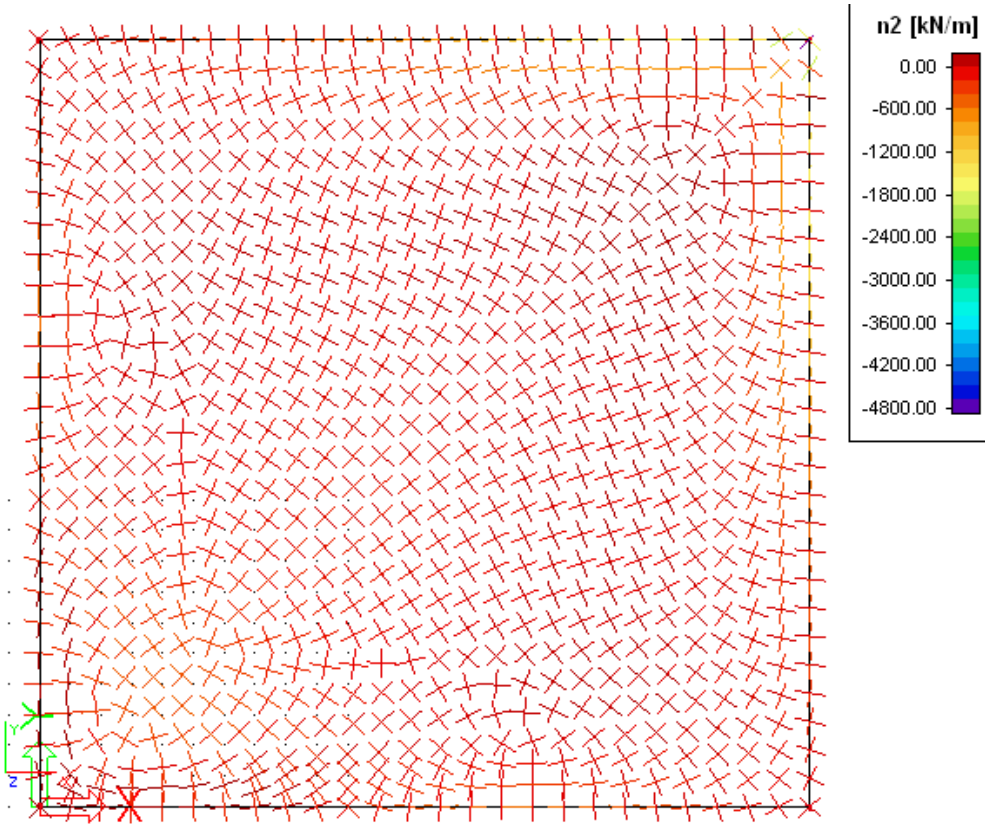


Figure 6. Principal directions of the normal forces

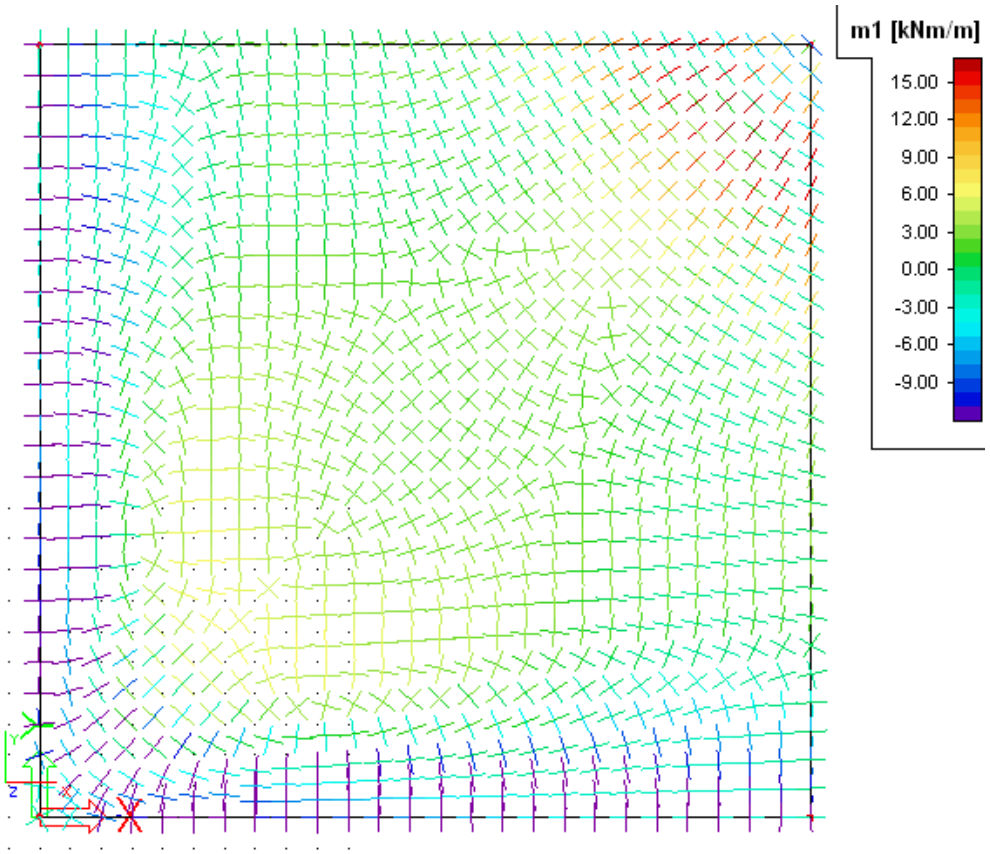


Figure 7. Principal directions of the moments

Exam CT4143, 1 July 2010
Solution to Problem 1

Sagitta …… Height of an arch
Compatibility equation ……

$$-\varepsilon_{xx,yy} + \gamma_{xy,xy} - \varepsilon_{yy,xx} = -k_y \kappa_{xx} + k_{xy} \rho_{xy} - k_x \kappa_{yy}$$

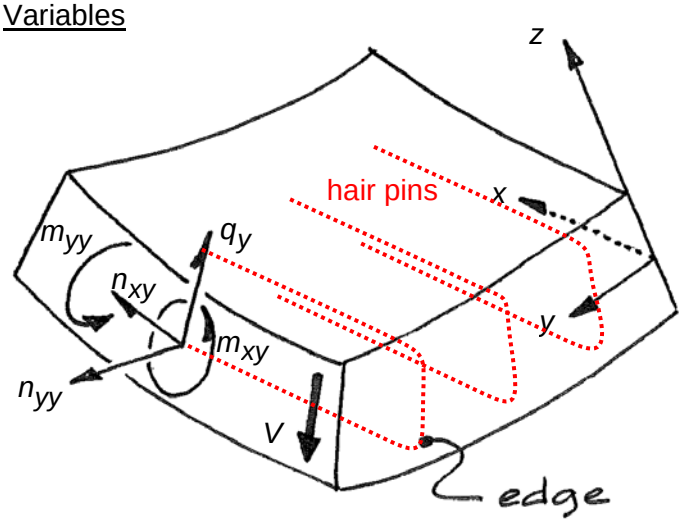
Equivalent stress or Von Mises stress ……..

$$\sigma_{VM} = \sqrt{\frac{1}{2}\left((\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2\right) + 3(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2)}$$

CNIT …….. Largest shell in the world, built in Paris
Principal directions …….. Eigenvectors of a second-order tensor, Largest and smallest value of curvature, moments, normal forces and stresses
Minimal surface …….. Surfaces with a zero mean curvature everywhere, for example a soap film
Critical load ……. Theoretical buckling load computed by eigenvalue analysis
Yoshimura pattern …..…….. Plastic buckling deformation of a shell

Solution to Problem 2

a Variables



b Shear stress

$$\sigma_{yz} = \frac{3}{2}\frac{q_y}{t} + 5\frac{m_{xy}}{t^2}$$

c Edge beams
To strengthen the shell (much of the force flow goes along the shell edges) and to prevent in-extensional deformation.

d Disadvantage
A large edge beam deforms differently than the shell to which it is attached. This causes compatibility moments, which can cause damage.

e Reinforcement
Hair pins (see figure above)

Solution to Problem 3

a A

b No body knows. Any answer is okay. (Just like to know your opinion.)

c A

d C

e B

Solution to Problem 4

a Gaussian curvature
During in-extensional deformation the strains of the middle surface do not change. Also, the derivatives of the strains to *x* and *y* do not change. Therefore the left-hand side of

$$-\varepsilon_{xx,yy} + \gamma_{xy,xy} - \varepsilon_{yy,xx} = -k_y \kappa_{xx} + k_{xy} \rho_{xy} - k_x \kappa_{yy}$$

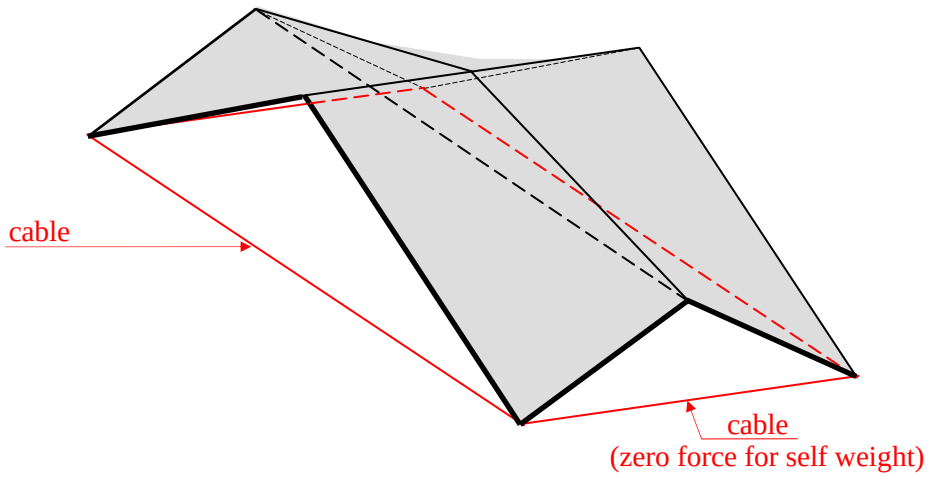
is zero. So, also the right-hand side is zero. The right-hand side is equal to the change of the Gaussian curvature (proved in handout 7c). Consequently, "during in-extensional deformation the Gaussian curvature does not change." Q.E.D.

b Example
Axially loaded cylinder.

c Spotting
Very large deformations indicate in-extensional deformation. Little change in Gaussian curvature confirms it.

Solution to Problem 5

a Cables
Cables are needed for equilibrium (see figure). This is a result of the solution to 5c.



b Membrane force and moment

The direction of the dead load is approximately perpendicular to the shell surface. Therefore the distributed membrane force is

$n_{xy} = \frac{1}{2} a p_z = \frac{1}{2} 27.3 \times 3 = 41.0$ kN/m.

The length of the long interior beams is $\sqrt{12^2 + 3^2} = 12.4$ m.

The length of the long edge beams is $\sqrt{12^2 + 6^2} = 13.4$ m.

The length of the short edge beams is $\sqrt{6^2 + 3^2} = 6.71$ m.

The largest moment in the shell occurs at the short interior beams. Rotation will not occur due to symmetry. The moment is

$$m_{xx} = 0.511 \frac{p}{l^2} \sqrt[3]{(a t l)^4} = 0.511 \frac{3}{6^2} \sqrt[3]{(27.3 \times 0.040 \times 6)^4} = 0.522 \text{ kNm/m.}$$

c Force flow

The normal force in the short interior beams is $2 \times 41 \times 6 = 491$ kN compression.
The normal force in the long interior beams is $2 \times 41 \times 12.4 = 1013$ kN tension.
The horizontal and vertical components of this force are

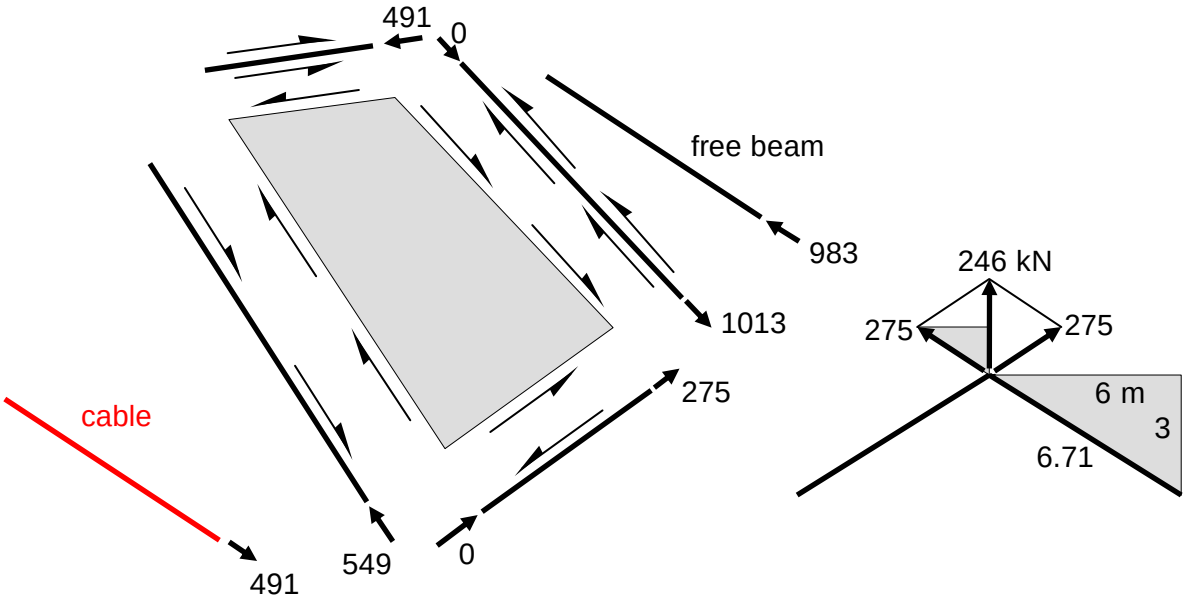
$N_h = 1013 \frac{12}{12.4} = 983$ kN , which is carried by the free beam.

$N_v = 1013 \frac{3}{12.4} = 246$ kN, which is carried by the short edge beams.

The force that goes into one short edge beam is

$\frac{1}{2} 246 \frac{6.71}{3} = 275$ kN tension

The normal force in the short edge beams is $275 - 41 \times 6.71 = 0$ kN.
The normal force in the long edge beams is $41 \times 13.4 = 549$ kN compression.



The horizontal and vertical components are

$$N_h = 549\frac{12}{13.4} = 491\text{kN}, \qquad N_v = 549\frac{6}{13.4} = 246\,\text{kN}.$$

The cable force is 491 kN tension.
The support reaction is $R = 246 + 0 = 246$ kN.

As a check we calculate the load on one hypar $G = 3 \times \frac{6 + 6.71}{2} \times \frac{12.3 + 13.4}{2} = 246$ kN,

which is equal to the support reaction.

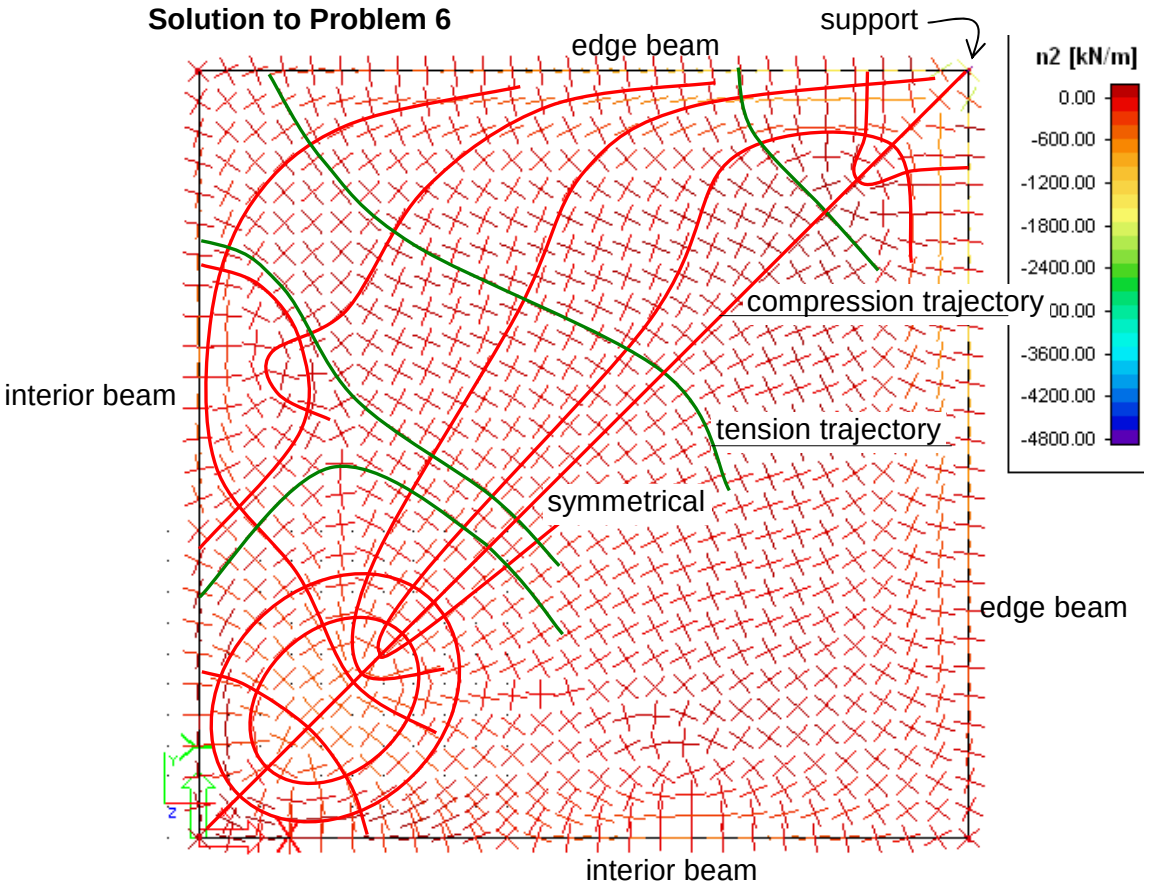
d Buckling

The critical membrane force is

$$n_{xy,cr} = 0.6\frac{E\,t^2}{a} = 0.6\frac{15 \times 10^6 \times 0.040^2}{27.3} = 527 \text{ kN/m}$$

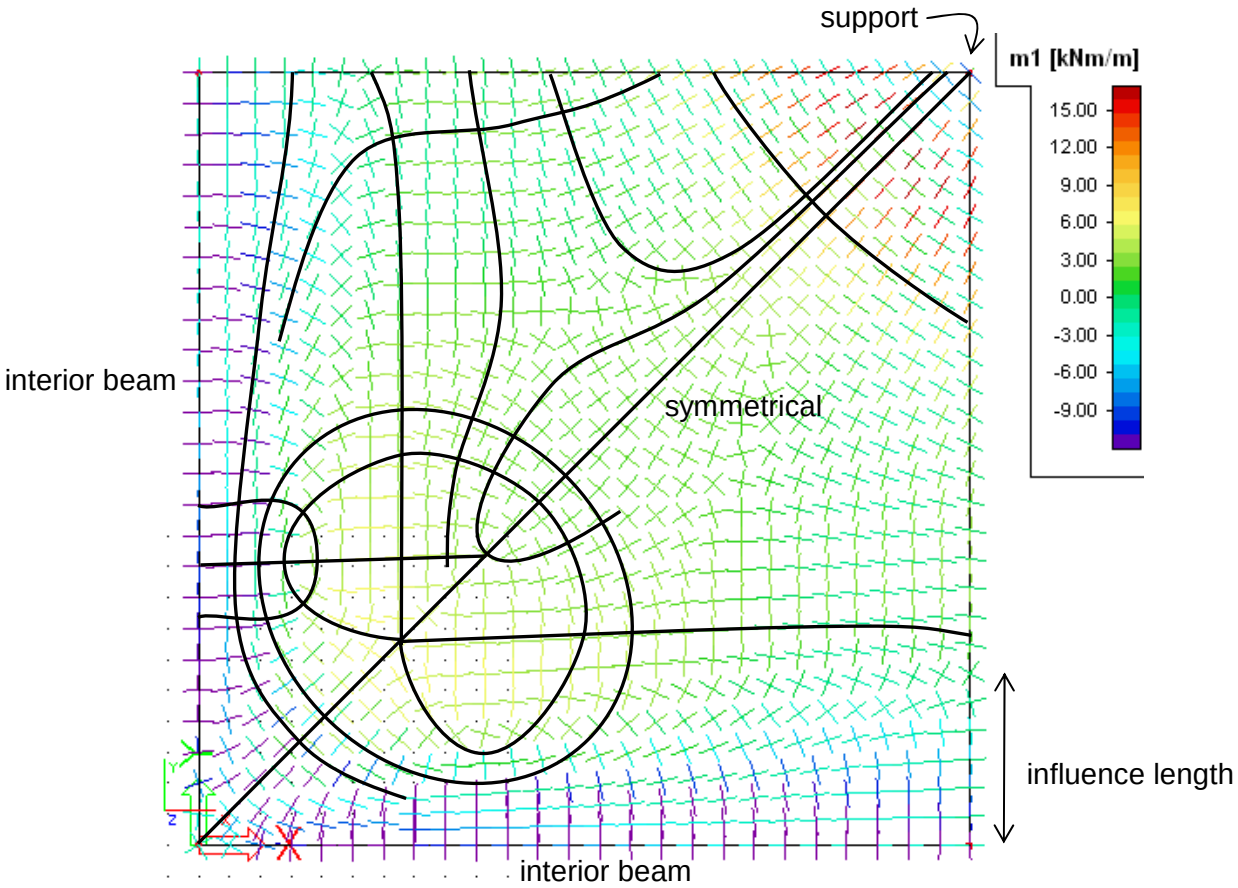
(A knockdown factor is not needed because hypars are not sensitive to imperfections.)
The actual membrane force (41 kN/m) is much smaller, therefore, it will not buckle.

Solution to Problem 6



Normal forces trajectories

In the top of the shell (origin) the force flow is complicated. Nevertheless, the trajectories show how the load is carried to the support. Much of the load goes into the edge beams. Large normal forces occur in the edges of the shell close to the support.



Moment trajectories

The moment trajectories are complicated too. Large moments occur at the interior beams. This is because there no rotation can occur due to symmetry. They flow into the shell perpendicular to the edges. At a short distance (influence length) they have almost vanished. Even larger moments occur near the support. The moments are compatibility moments because the deformation of the thin shell is constrained by the interior beams and the support.