Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CIE4143 Shell Analysis
Tuesday 15 August 2017, 13:30 – 16:30 hours

Problem 1

Explain shortly the meaning of the following eight words in relation to shells structures.

Membrane forces
Edge beam
Sanders-Koiter equations
Knockdown factor
Gaussian curvature
Elpar
Love
Abutment force

Problem 2

a Figure 1 shows part of a shell and edge beam with coordinate system. Draw in this figure the shell loading, membrane forces, moments and shear forces in the positive directions. (You can hand in this page with your answers.)

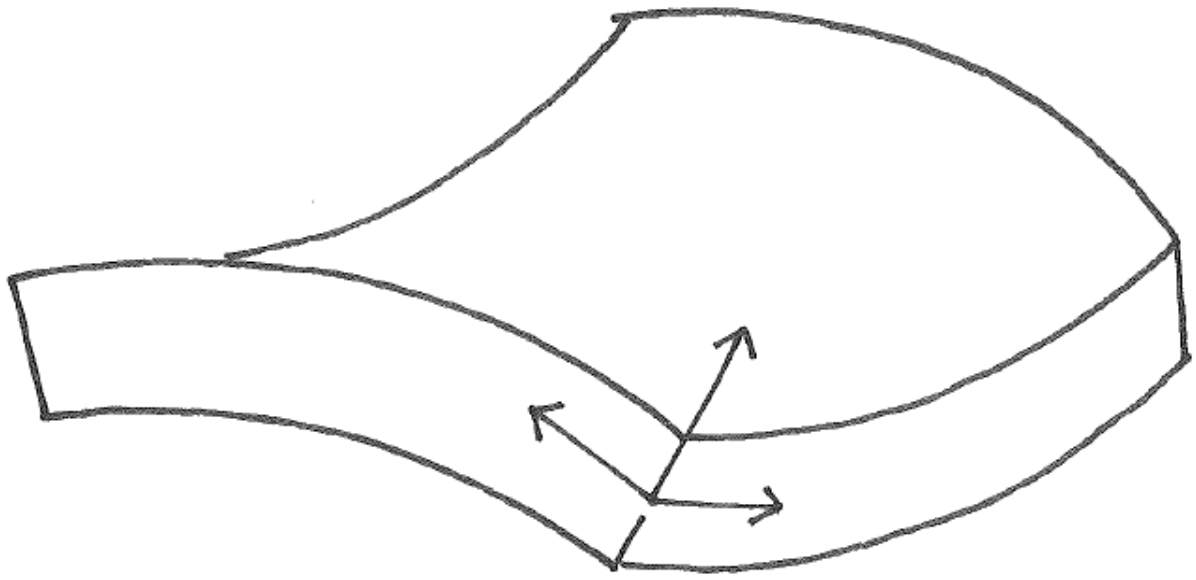


Figure 1. Part of a shell

b Does k_{xx} have a positive value or a negative value? (fig. 1)
Does k_{yy} have a positive value or a negative value? (fig. 1)
Does k_G have a positive value or a negative value? (fig. 1)

Problem 3

a Which variables are in the Sanders-Koiter equations? Choose A, B, C or D.

A shape $\bar{x}(u,v), \bar{y}(u,v), \bar{z}(u,v)$

B slopes $\frac{\partial z}{\partial x}(u,v)$, $\frac{\partial z}{\partial y}(u,v)$

C curvatures $k_{xx}(u,v), k_{yy}(u,v), k_{xy}(u,v)$

D Lame parameters $\alpha_x(u,v), \alpha_y(u,v)$

Consider a Taylor expansion of k_{xx} in $u = v = 0$.

$$k_{xx} = b_1 + b_2 u + b_3 v + b_4 u^2 + b_5 uv + b_6 v^2 + b_7 u^3 + b_8 u^2 v + b_9 uv^2 + b_{10} v^3$$

b Which constants will end up in the solution of p_z? Choose A, B, C or D.

A b_1 to b_{10}

B b_2 to b_{10}

C b_4 to b_{10}

D b_7 to b_{10}

c We want to make the expansion of k_{xx} as short as possible. We also want to compute the exact value of p_z <u>in point $u = v = 0$</u>. Which constants should be included in the expansion? Choose A, B, C or D.

A b_1

B b_1 to b_3

C b_1 to b_6

D b_7 to b_{10}

d Consider two shell parts that are connected in the line $u = 0$. The curvatures of the parts have Taylor expansions with the same values of b_1 and b_2. The other constants in the Taylor expansions are different. What is the level of continuity of the connection? Choose A, B, C or D.

A C_0 continuity

B C_1 continuity

C C_2 continuity

D C_3 continuity

e What level of continuity in the shape is required to prevent <u>any</u> edge disturbance?

A C_0 continuity

B C_1 continuity

C C_2 continuity

D C_5 continuity

f What is the level of continuity between shell finite elements?

A C_0 continuity
B C_1 continuity
C C_2 continuity
D C_5 continuity

Consider a Taylor expansion of α_x in $u = v = 0$.

$$\alpha_x = b_1 + b_2 u + b_3 v + b_4 u^2 + b_5 uv + b_6 v^2 + b_7 u^3 + b_8 u^2 v + b_9 uv^2 + b_{10} v^3 .$$

g Which constants will end up in the solution of p_z? Choose A, B, C or D.

A b_1 to b_{10}
B b_2 to b_{10}
C b_4 to b_{10}
D b_7 to b_{10}

h We want to make the expansion of α_x as short as possible. We also want to determine the exact value of p_z <u>in point $u = v = 0$</u>. Which constants should be included?
Choose A, B, C or D.

A b_1
B b_1 to b_3
C b_1 to b_6
D b_7 to b_{10}

i Consider improving the Sanders-Koiter equations with geometrical nonlinearity (large deformations). In theory, how can this be done? Choose A, B, C or D.

A Replace $\bar{x}, \bar{y}, \bar{z}$ by $\bar{x} + u_x, \bar{y} + u_y, \bar{z} + u_z$.
B Replace k_{xx}, k_{yy}, k_{xy} by $k_{xx} - \kappa_{xx}, k_{yy} - \kappa_{yy}, k_{xy} - \frac{1}{2}\rho_{xy}$.
C Replace k_{xx}, k_{yy}, k_{xy} by $k_{xx} + \kappa_{xx}, \quad k_{yy} + \kappa_{yy}, \quad k_{xy} + \rho_{xy}$.
D Replace k_{xx}, k_{yy}, k_{xy} by $\kappa_{xx}, \kappa_{yy}, \rho_{xy}$.

Problem 4

A shell structure has the shape of halve a torus (fig. 2). The diameter is 2 m. The diameter of the torus centre line is 8 m. The shell wall is 4 mm thick. One edge is fixed. A point load of 75 kN is applied to the free edge in the positive *z* direction. The principal membrane forces are shown in figure 3. Shear deformation has not been included in the computation (Kirchhoff instead of Mindlin-Reissner).

diameter of the torus centre line	8000 mm
small diameter of the torus	2000 mm
thickness	t = 4 mm
Young's modulus	$E = 2\ 10^5$ N/mm^2
Poisson's ratio	$\nu = 0.2$
mass density	$\rho = 7850$ kg/m^3
yield strength	500 N/mm^2
gravitational acceleration	$g = 9.8$ m/s^2
element size	100 mm x 100 mm
load	F = 75 kN
theoretical buckling force	$n_{cr} = 0.6\ E\ t^2/a$
knockdown factor	C = 1/6

a Is this shell thin? Explain your answer.

b Calculate the influence length.

c Are the elements sufficiently small? Explain your answer.

d Will this shell buckle? Explain your answer.

e What other ways exist to check the buckling strength?

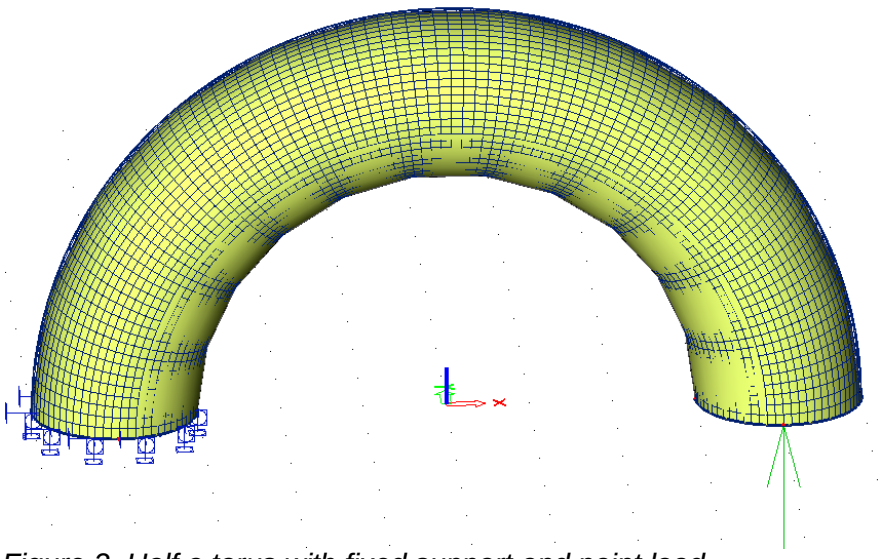


Figure 2. Half a torus with fixed support and point load

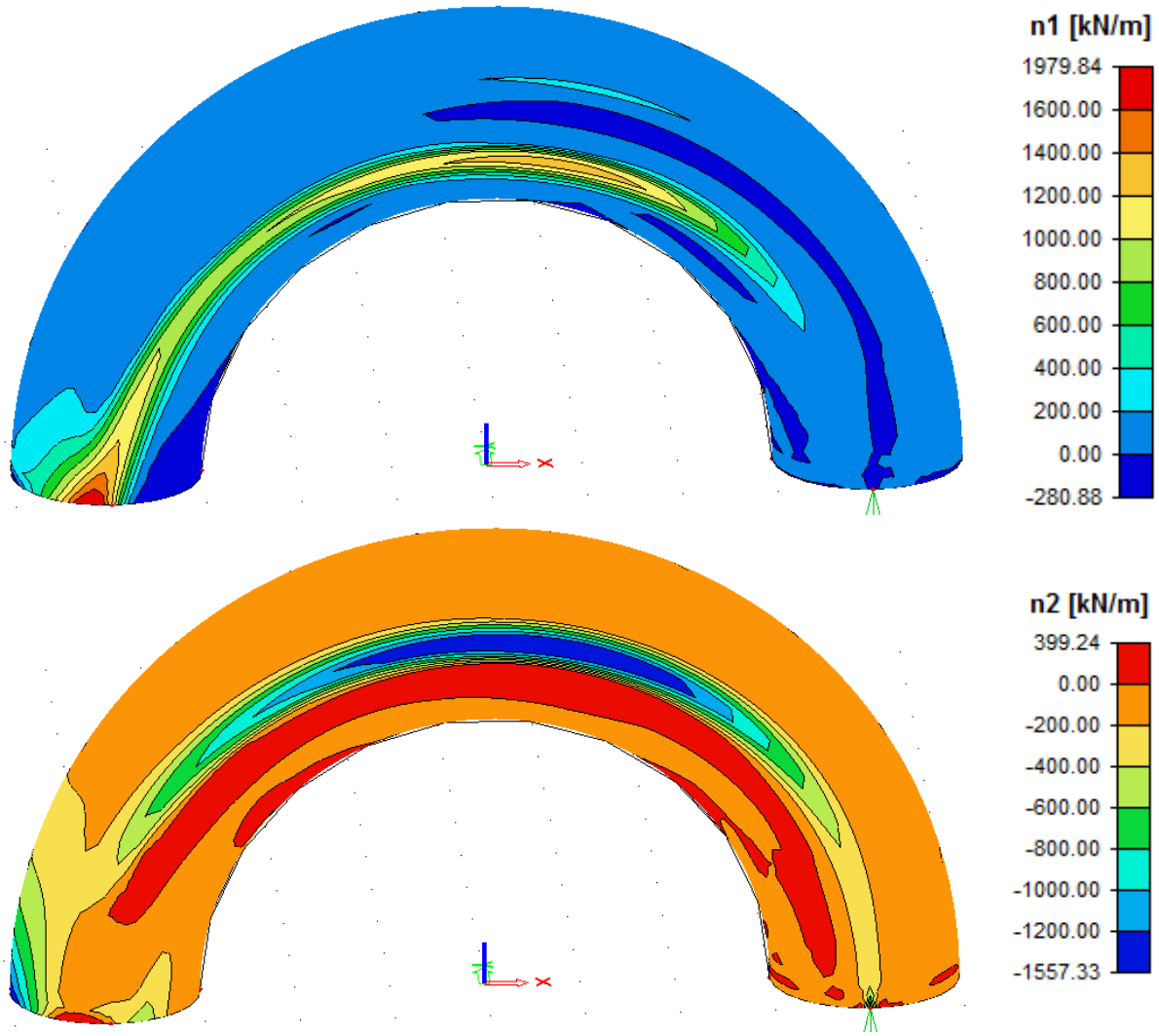


Figure 3. Principal membrane forces in the torus

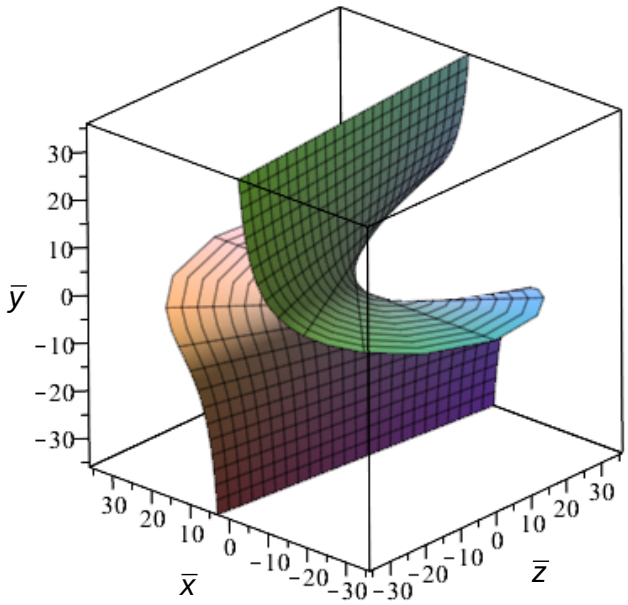


Figure 4. Shell shape, a = 3, -12 < *u* < 12, -12 < *v* < 12,

Problem 5

Consider the parameterisation (fig. 4)

$$\bar{x} = u\,a\left(1 - \frac{v^2}{1+v^2+\sqrt{1+v^2}}\right)$$
$$\bar{y} = v\,a$$
$$\bar{z} = \frac{\bar{x}\,\bar{y}}{a}$$

a What is the name of this shape?

b What is the physical meaning of parameter *a*?
What is the unit of a?
What are the units of *u* and *v*?

c Is this parameterisation orthogonal? Explain your answer. (You can use Maple or another mathematical program.)

Maple has been used to calculate Lame's parameters and the curvatures of this shape. These equations are not provided because they are large. Instead the Taylor expansions at $u = v = 0$ are provided.

$$\alpha_x = a \qquad \alpha_y = a\left(1 + \tfrac{1}{2}u^2 - u^2v^2 - \tfrac{1}{8}u^4 + \ldots\right)$$

$$k_{xx} = 0 \qquad k_{yy} = \frac{1}{a}(-2uv + \ldots) \qquad k_{xy} = \frac{1}{a}(1 - u^2 - v^2 + \ldots)$$

$$k_1 = \frac{1}{a}(1 - u^2 - uv - v^2 + \ldots) \qquad k_2 = \frac{1}{a}(-1 + u^2 - uv + v^2 + \ldots)$$

$$k_m = \frac{1}{a}(-uv + \ldots) \qquad k_G = \frac{1}{a^2}(-1 + 2u^2 + 2v^2 - \ldots)$$

d Is the parameterisation in the principal curvature directions? Explain your answer.

e Check the provided Gaussian curvature.

f Is the deformation of this shape extensional? Choose A, B, C or D.

A Yes because the Gaussian curvature changes; it depends on *u* and *v*.
B Yes because the distance between the parameter lines varies.
C No because a flat sheet can be twisted in this shape without stretching.
D There is no deformation; it is just a shape.

g Check the equation of Gauß for this shape at $u = v = 0$.

Problem 6

In engineering we often can choose between calculating something and using a table. Let us apply this to finite element analysis of shell structures. A computer can compute an element stiffness matrix or it can look it up in a table. The question is; which costs less?

Consider a linear elastic shell element with 3 nodes and 6 dofs per node.
The following data applies.

Computation time costs 0.30 euro/hour (maintenance, electricity and write off).
Hard disk space costs 0.20 euro/GB/year.
In one year, on one computer, $3 \cdot 10^{10}$ times a shell element is computed or looked up.
Finding one number in a list takes $\log_2 n$ evaluations of "<", where n is the number of steps in the domain of the input variable.
The domain of each input variable of the table is divided in 10 steps.
A computer can perform 10^{10} evaluations of "<" per second.
Computing a 3 node shell element stiffness matrix takes 4000 multiplications or additions.
A computer can perform 10^{10} multiplications and additions per second.

a Which input variables should the table have? For example shell thickness, …
How many input variables have you selected?

b How many output variables (numbers) should the table have? Explain your answer.

c Give smart ways to reduce the number of input and output variables of the table.

d Suppose that the thicknesses of shell elements vary between 0.01 mm and 1 m.
For the table we need to divide this domain in steps.
(Proposed are 10 steps. This is an estimation.)
What determines the actual step size?
How can a computer proceed to determine the correct step size?

e How many numbers need be stored in the table?
How much GB is this?

f How much time does it take to find the right element stiffness matrix in the table?

g How much time does it take to compute an element stiffness matrix?

h Which costs less; computing or looking up in a table? Explain your answer.

i What has most influence on answer **h**?

Answers to problem 1

Membrane forces …….. in plane normal forces and shear forces $n_{xx}, n_{yy}, n_{xy}, n_{yx}$

Edge beam …….. beam at the edge of a thin shell

Sanders-Koiter equations … 21 equations that describe the behaviour of thin shells

Knockdown factor ……. reduction of the theoretical linear buckling load to include the effect of imperfections

Gaussian curvature …….. $k_1 k_2$

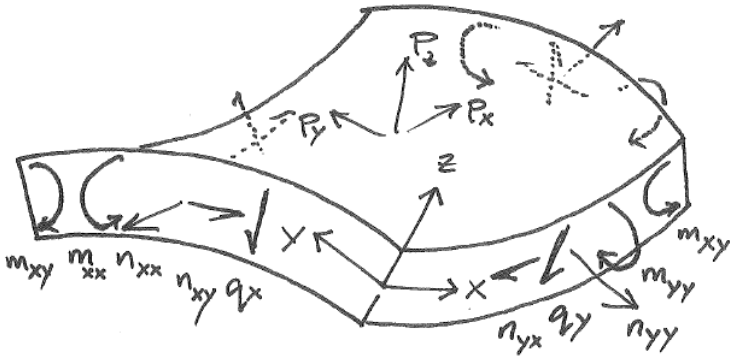
Elpar …….. elliptical paraboloid

Love …….. Augustus Love was the first to propose a shell theory.

Abutment force …….. horizontal load on the support of an arch

Answers to problem 2

a



b k_{xx} is positive. k_{yy} is negative. k_G is negative.

Answers to problem 3

a C
b A
c C
d D
e D
f A
g A
h C
i B

Answer to problem 4

a $a / t = 1000 / 4 = 250 > 30$ So it is thin.

b $2.4\sqrt{at} = 2.4\sqrt{1000 \times 4}$ = 152 mm

c The elements are too large at the edges. About 6 should be in the influence length.

d $n_{cr} = 0.6\dfrac{E\,t^2}{a} = 0.6\dfrac{2 \cdot 10^5\ 4^2}{1000}$ = 1920 N/mm = 1920 kN/m

$n_{ult} = \dfrac{n_{cr}}{6}$ = 320 kN/m < 1557 kN/m

It will buckle.

e - Perform a linear buck buckling analysis with the software
(and apply the knock down factor).
- Add imperfections and perform a static nonlinear analyses.

Answers to problem 5

a Hypar
(The figure appears strange because of the edges created by this particular parameterisation and the very large domain of *u* and *v*. Nevertheless, the equations show that this is a hypar.)

b Radius of curvature in the origin. *a* has the unit length, for example m. *u* and *v* have no unit.

c Substitution in $\frac{\partial \overline{x}}{\partial u}\frac{\partial \overline{x}}{\partial v}+\frac{\partial \overline{y}}{\partial u}\frac{\partial \overline{y}}{\partial v}+\frac{\partial \overline{z}}{\partial u}\frac{\partial \overline{z}}{\partial v}$ gives zero. So it is orthogonal.

d No because $k_{xy} \neq 0$.

e $k_G = k_1 k_2 = \frac{1}{a}(1-u^2-uv-v^2+...)\frac{1}{a}(-1+u^2-uv+v^2+...) =$

$\frac{1}{a^2}(-1+u^2-uv+v^2+u^2-u^4+u^3v-u^2v^2+uv-u^3v+u^2v^2-uv^3+v^2-u^2v^2+uv^3-v^4+...) =$

$\frac{1}{a^2}(-1+2u^2+2v^2-u^2v^2-u^4-v^4+...) =$

$\frac{1}{a^2}(-1+2u^2+2v^2-...)$ *correct*

f D

g

$$k_G = \frac{-1}{\alpha_x \alpha_y}\left[\frac{\partial}{\partial u}\left(\frac{1}{\alpha_x}\frac{\partial \alpha_y}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\alpha_y}\frac{\partial \alpha_x}{\partial v}\right)\right]$$

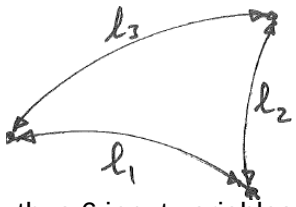
$$\frac{1}{a^2}(-1+2u^2+2v^2+...) =$$

$$= \frac{-1}{a^2(1+\frac{1}{2}u^2-u^2v^2-\frac{1}{8}u^4+...)}\left[\frac{\partial}{\partial u}\left(\frac{1}{a}\frac{\partial(a(1+\frac{1}{2}u^2-u^2v^2-\frac{1}{8}u^4+...))}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{a(1+\frac{1}{2}u^2-u^2v^2-\frac{1}{8}u^4+...)}\frac{\partial a}{\partial v}\right)\right]$$

$$= \frac{-1}{a^2(1+\frac{1}{2}u^2-u^2v^2-\frac{1}{8}u^4+...)}\left[\frac{\partial}{\partial u}\left(\frac{1}{a}a(u-2uv^2-\frac{1}{2}u^3...)\right)+\frac{\partial}{\partial v}\left(\frac{1}{a(1+\frac{1}{2}u^2-u^2v^2-\frac{1}{8}u^4+...)}0\right)\right]$$

$$= \frac{-1}{a^2(1+\frac{1}{2}u^2-u^2v^2-\frac{1}{8}u^4+...)}\left[1-2v^2-\frac{3}{2}u^2+...\right] = \frac{1}{a^2}(-1+2u^2+2v^2+...)$$

Answers to problem 6



a $l_1, l_2, l_3, k_{xx}, k_{yy}, k_{xy}, t, E, \nu$
These are 9 input variables.
(If you considered a flat element, thus 6 input variables, then this would be correct too.)

b Output variables: A stiffness matrix has $(3 \times 6)^2 = 324$ elements.

c Key words:
- Stiffness is linear in *E.*
- Dimensionless parameters
- Eliminate element rigid body modes
- Symmetry of the stiffness matrix

d Keywords:
- Accuracy of the output
- Interpolation

For example, a computer can compute for small steps and disregard the step results for which the difference with an interpolation is small.

e $(10+1)^9 \times 324 = 8 \cdot 10^{11}$
$8 \cdot 10^{11} \times 8 = 6 \cdot 10^{12}$ byte = 6000 GB

f $\log_2 10 \times 9 / 10^{10} = 3 \cdot 10^{-10}$ s

g $4000 / 10^{10} = 4 \cdot 10^{-7}$ s

h Computing: $3 \cdot 10^{10} \times 4 \cdot 10^{-7} / 3600 \times 0.30 = 1$ euro/year
Looking up: $6000 \times 0.20 + 3 \cdot 10^{10} \times 3 \cdot 10^{-10} / 3600 \times 0.30 = 1200 + 0.001 = 1200$ euro/year
Computing costs less.

i Hard disk space