Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your <u>name</u> and <u>study number</u> at the top right-hand of your work.

Exam CT4150 Plasticity Theory
Wednesday 22 August 2006, 14:00 – 17:00 hours

Problem 1

A frame consists of two columns with strength $4M_p$ and 12 beams with strength M_p (Fig. 1). The columns are rigidly connected to the beams and to the foundation. The frame is loaded by an evenly distributed load q. The following relation exists between the plastic moment M_p and the plastic normal force N_p (Fig. 2).

$$N_p = \beta \frac{M_p}{a}$$

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

a Assume $\beta \to \infty$. Determine the collapse load q for important mechanisms. Write the collapse loads as functions of M_p and a. What is the decisive collapse load? (1.5 points)

b Assume $\beta \to \infty$. Draw the bending moment diagram and calculate the support reactions for the structure at the moment of collapse. (1 point)

c Assume $\beta = 96$. Choose one of the following problems (You need not do both).

– Determine the largest <u>lower-bound</u> for q.

– Determine the smallest <u>upper-bound</u> for q.

Use the decisive mechanism of problem **1a** (2 points).

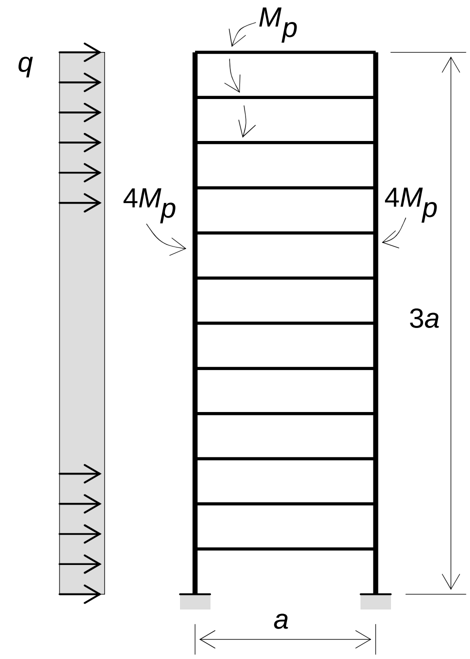


Figure 1. Frame

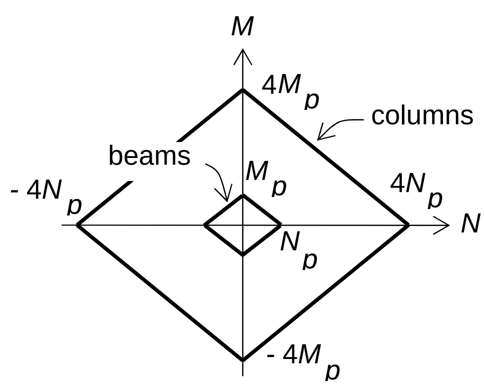


Figure 2. Interaction diagram

Problem 2

A plate with an opening is simply supported at three edges (Figure 3). The plate carries an evenly distributed load q [kN/m²]. The plate is homogeneous. Reinforcing bars in the bottom of the plate provide a positive yield moment $2m_p$ [kNm/m] in the y direction and m_p in the x direction. The top face of the plate is not reinforced.

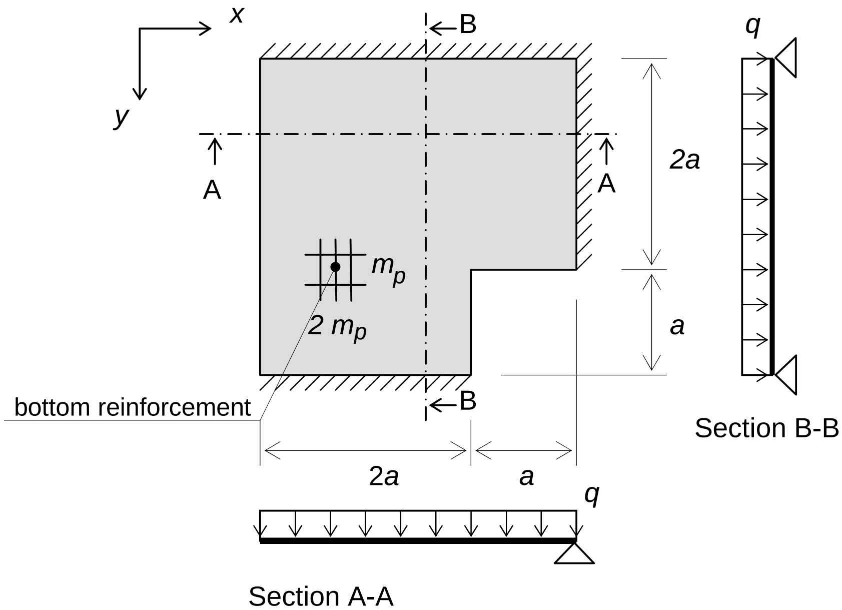


Figure 3. Square plate with an opening

a We consider the yield line patterns of Figure 4. Which of these patterns give kinematically possible mechanisms (1 point).

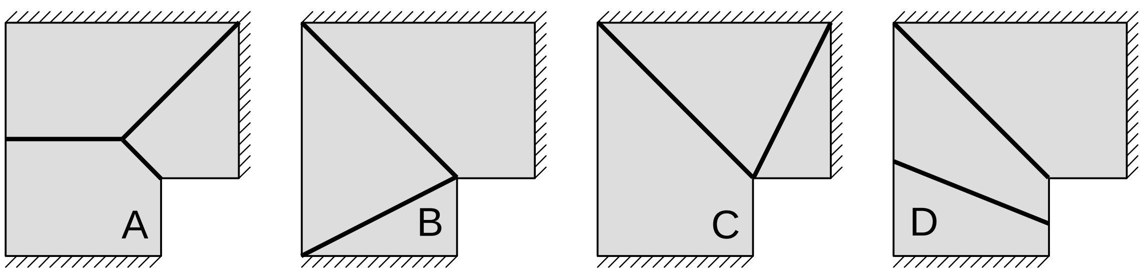


Figure 4. Yield line patterns of problem 2a

b We consider the yield line pattern of Figure 5. Determine an <u>upper bound</u> for q expressed in m_p and a (1.5 points).

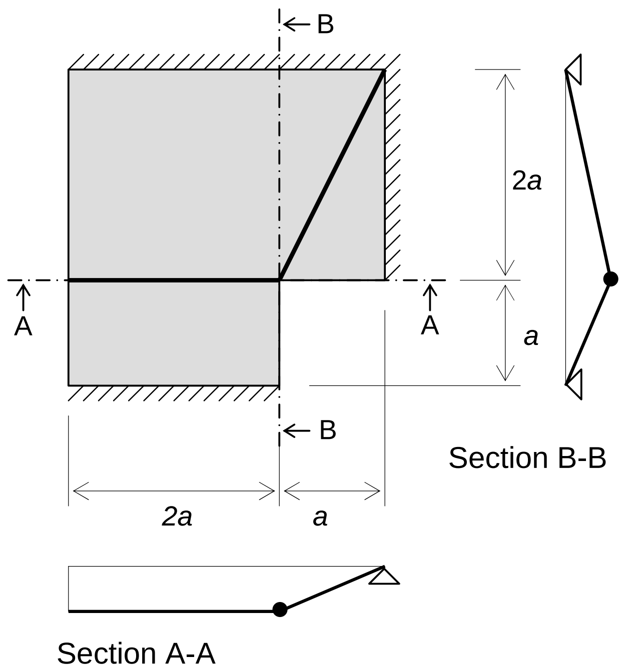


Figure 5. Yield line pattern of problem 2b

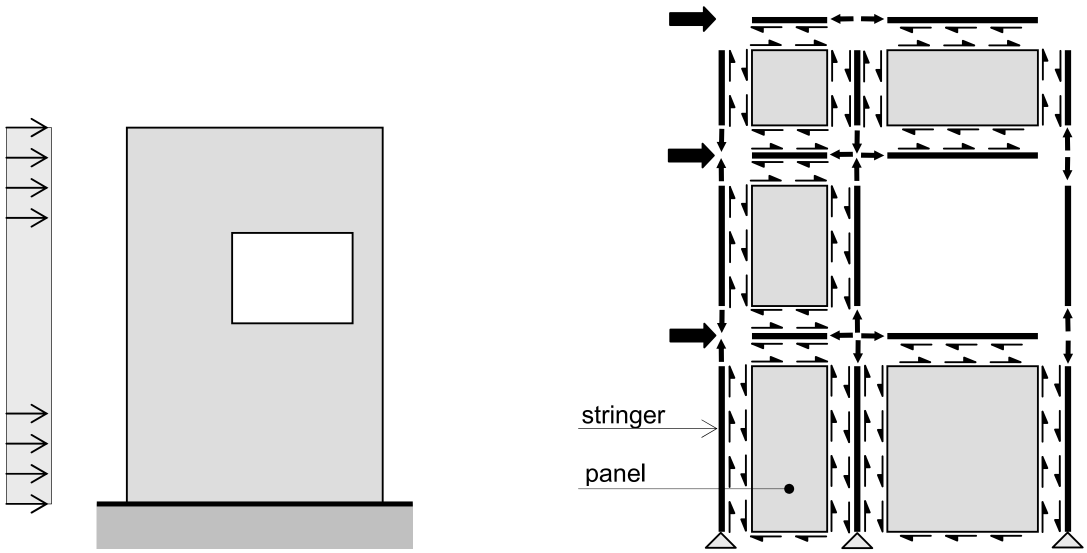
c Determine the largest lower bound for q using torsion free beams ($m_{xy} = 0$) in the x direction and y direction (1.5 points).

Problem 3

a Upper bounds and lower bounds for ultimate loading of plates often have large differences. What causes these differences? (0.5 point)

b Engineers in Denmark often use the “stringer method” to calculate the ultimate load of reinforced concrete walls [1]. In this method a wall is modelled as an equilibrium system of stringers that carry only axial forces and panels that carry only shear forces (Fig. 6). The optimal force flow is determined by a computer. Is the stringer method an upper bound analysis or a lower bound analysis? (0.5 points)

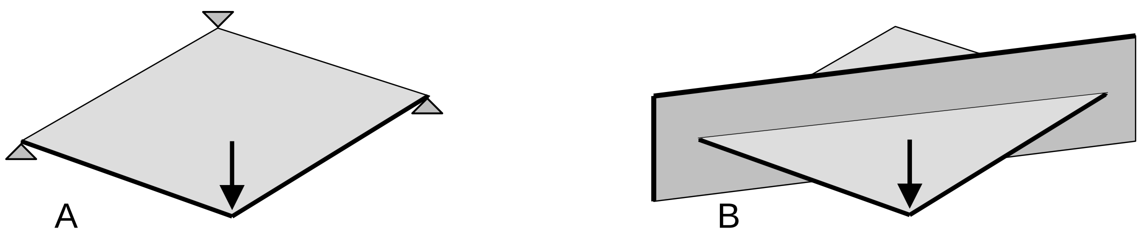
c A square steel plate is loaded by a force in one of the corners (Fig. 7). Which of the support conditions – A or B – gives the largest collapse load? (0.5 point)



Wall with opening and loading

Stringer-panel model of the wall

Figure 6. Stringer method



Supported at 3 corners

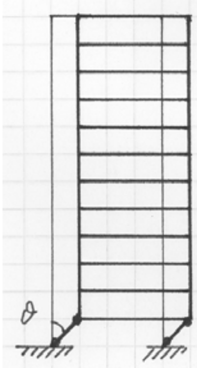
Clamped at a diagonal

Figure 7. Square steel plate loaded in one corner

Literature

[1] M.P. Nielsen, "Limit Analysis and Concrete Plasticity", ISBN 0 13 536623 2, Prentice-Hall, London, 1984.

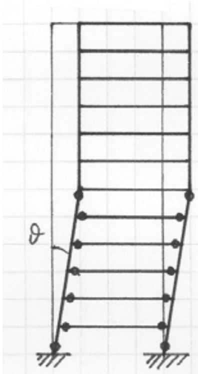
Answer to Problem 1a



$$A = q\,\tfrac{11}{4}a\,\vartheta\,\tfrac{1}{4}a + q\,\tfrac{1}{4}a\,\vartheta\,\tfrac{1}{8}a = \tfrac{23}{32}\,q\,a^2\vartheta$$

$$E = 4M_p\,\vartheta\,4 = 16M_p\vartheta$$

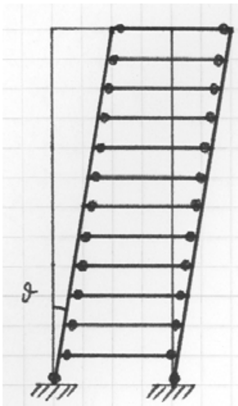
$$A=E \;\Rightarrow\; q = \frac{512}{23}\frac{M_p}{a^2} = 22{,}26\,\frac{M_p}{a^2}$$



$$A = q\,\tfrac{6}{4}a\,\vartheta\,\tfrac{6}{4}a + q\,\tfrac{6}{4}a\,\vartheta\,\tfrac{6}{8}a = \tfrac{27}{8}\,q\,a^2\vartheta$$

$$E = 4M_p\,\vartheta\,4 + M_p\,\vartheta\,10 = 26M_p\vartheta$$

$$A=E \;\Rightarrow\; q = \frac{208}{27}\frac{M_p}{a^2} = 7.70\,\frac{M_p}{a^2}$$

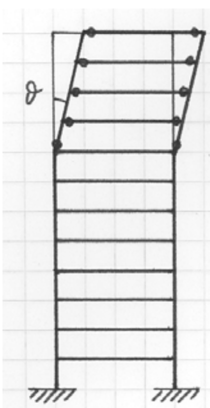


$$A = q\,3a\,\vartheta\,\tfrac{3}{2}a = \tfrac{9}{2}\,q\,a^2\vartheta$$

$$E = 4M_p\,\vartheta\,2 + M_p\,\vartheta\,24 = 32M_p\vartheta$$

$$A=E \;\Rightarrow\; \boxed{q = \frac{64}{9}\frac{M_p}{a^2}} = 7.11\,\frac{M_p}{a^2}$$

decisive

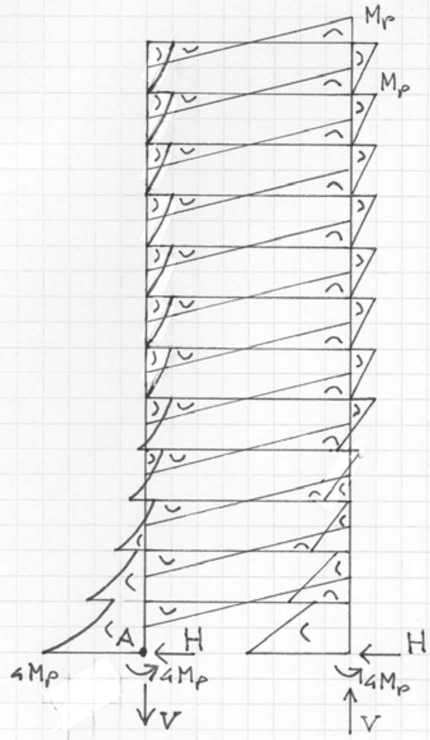


$$A = q\,a\,\vartheta\,\tfrac{1}{2}a$$

$$E = 4M_p\,\vartheta\,2 + M_p\,\vartheta\,8 = 16M_p\vartheta$$

$$A=E \;\Rightarrow\; q = 32\,\frac{M_p}{a^2}$$

Answer to Problem 1b



mom. A ↻

$$q\,3a\,\tfrac{3}{2}a - Va - 4M_p - 4M_p = 0$$

$$V = \tfrac{9}{2}qa - 8\frac{M_p}{a}$$

$$= \tfrac{9}{2}\tfrac{64}{9}\frac{M_p}{a} - 8\frac{M_p}{a} = 24\frac{M_p}{a}$$

hor. →

$$q\,3a - 2H = 0$$

$$H = \tfrac{3}{2}qa = \tfrac{3}{2}\tfrac{64}{9}\frac{M_p}{a} = \tfrac{32}{3}\frac{M_p}{a}$$

Answer to problem 1c Lower-bound

We reduce the loading with a factor α. Also the moments and the normal forces are reduced with α. We consider the columns because they are more overloaded than the beams. In a section with a moment $\alpha\,4M_p$ we can allow a normal force $(1-\alpha)\,4N_p$. Therefore, in the columns we have

$$N = \alpha 24\frac{M_p}{a} = (1-\alpha)\,4N_p = (1-\alpha)4\beta\frac{M_p}{a}\,.$$

Consequently,

$$\alpha 24 = (1-\alpha)4\beta$$

and

$$\alpha = \frac{\beta}{6+\beta}\,.$$

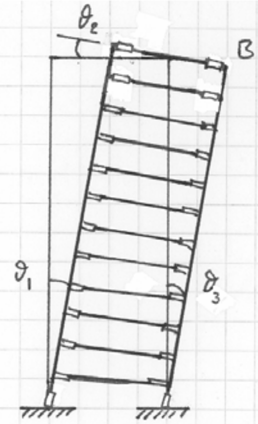
Since β = 96 we find

$$\alpha = \tfrac{16}{17}\,.$$

The collapse load is

$$q = \alpha\,\tfrac{64}{9}\frac{M_p}{a} = \tfrac{16}{17}\,\tfrac{64}{9}\frac{M_p}{a} = \tfrac{1024}{153}\frac{M_p}{a} = 6.69\frac{M_p}{a}$$

Answer to Problem 1c Upper-bound



$$u = \frac{a}{\beta}\vartheta \qquad \beta = 96$$

displacement of B $\rightarrow$

$$\vartheta_1 3a - (\vartheta_1 - \vartheta_2)\frac{a}{\beta} - (\vartheta_3 - \vartheta_2)\frac{a}{\beta} = \vartheta_3 3a$$

displacement of B $\downarrow$

$$-\vartheta_1 \frac{a}{\beta} + \vartheta_2 a = \vartheta_3 \frac{a}{\beta}$$

$$\begin{cases} \vartheta_2 = \dfrac{288}{13871}\vartheta_1 \\ \vartheta_3 = \dfrac{13777}{13871}\vartheta_1 \end{cases}$$

$$A = q\, 3a\, \vartheta_1 \tfrac{3}{2} a = \tfrac{9}{2} q a^2 \vartheta_1$$

$$E = 4 M_p \vartheta_1 + 4 M_p \vartheta_3 + M_p (\vartheta_1 - \vartheta_2)\, 12 + M_p (\vartheta_3 - \vartheta_2)\, 12$$

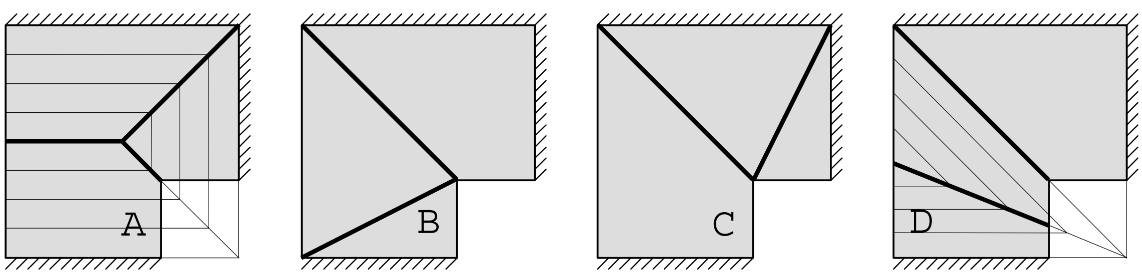
$$= M_p \vartheta_1 \left(4 + 4\frac{13777}{13871} + \left(1 - \frac{288}{13871}\right) 12 + \left(\frac{13777}{13871} - \frac{288}{13871}\right) 12\right)$$

$$= \frac{435456}{13871} M_p \vartheta_1$$

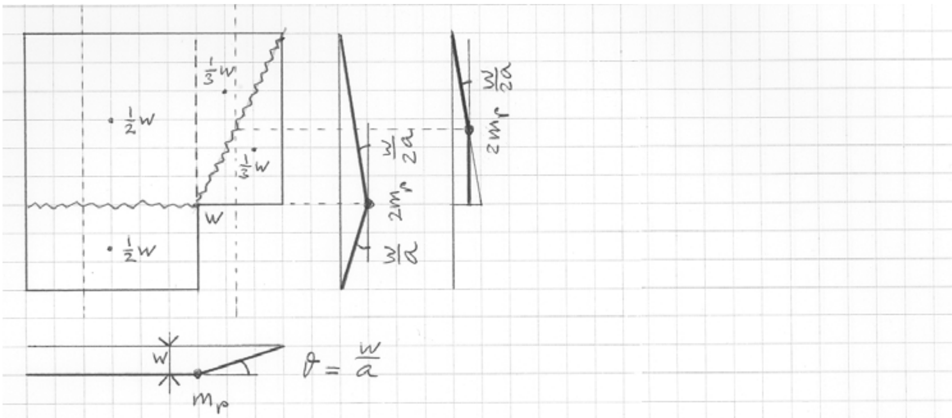
$$A = E \Rightarrow \quad q = \frac{96768}{13871}\frac{M_p}{a^2} = 6.97\frac{M_p}{a^2}$$

Answer to Problem 2a

Kinematically possible are patterns A and D. The figure below shows the altitude lines of the deformed mechanisms.



Answer to Problem 2b

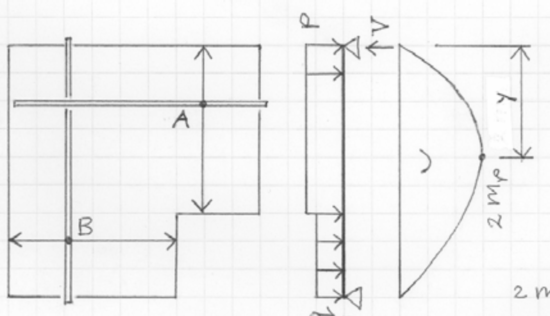


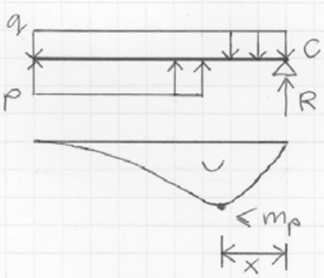
$$A = q\,2a\,2a\,\tfrac{1}{2}w + q\,2a\,a\,\tfrac{1}{2}w + q\,\frac{2a\cdot a}{2}\,\tfrac{1}{3}w + q\,\frac{2a\,a}{2}\,\tfrac{1}{3}w = \frac{11}{3}\,q a^2 w$$

$$E = 2m_p\,2a\left(\frac{w}{a} + \frac{w}{2a}\right) + 2m_p\,a\,\frac{w}{2a} + m_p\,2a\,\frac{w}{a} = 9\,m_p w$$

$$A = E \Rightarrow \boxed{q = \frac{27}{11}\,\frac{m_p}{a^2}} = 2.45\,\frac{m_p}{a^2}$$

Answer to Problem 2c





mom. D

$$p\,2a\,2a - V\,3a + q\,a\,\tfrac{1}{2}a = 0 \Rightarrow V = \frac{5}{3}qa$$

vert. ↓

$$py - V = 0 \Rightarrow y = \frac{40}{27}a$$

$$2m_p = Vy - p\,y\,\tfrac{1}{2}y = \frac{5}{3}qa\,\frac{40}{27}a - \frac{9}{8}q\,\frac{1}{2}\left(\frac{40}{27}a\right)^2$$

$$\boxed{q = \frac{81}{50}\,\frac{m_p}{a^2}} = 1.62\,\frac{m_p}{a^2}$$

mom. C

$$q\,3a\,\tfrac{3}{2}a - p\,2a\,2a = 0 \Rightarrow p = \frac{9}{8}q$$

vert. ↓

$$q\,3a - p\,2a - R = 0 \Rightarrow R = \frac{3}{4}qa$$

$$qx - R = 0 \Rightarrow x = \frac{3}{4}a$$

$$M = -qx\,\tfrac{1}{2}x + Rx = \frac{9}{32}\,qa^2 = \frac{729}{1600}\,m_p < m_p$$

O.K.

Answer to Problem 3a

The difference is for a large part due to neglecting the torsion strength in the lower bound solution.

Answer to Problem 3b

The stringer method is a lower bound analysis.

Answer to Problem 3c

For steel the Von Mises yield contour needs to be used.

The ultimate load of situation A is $F = \frac{2}{\sqrt{3}} m_p = 1.15 m_p$. (lecture book. plate part, p. 59)

The ultimate load of situation B is $F = 2m_p$. (lecture book, plate part, p. 50)

Therefore, situation B gives a higher collapse load.