Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT4150 Plasticity Theory
Thursday 18 January 2007, 9:00 – 12:00 hours

Problem 1

A frame consists of members with a strength $5M_p$ and a column with a strength M_p (Fig. 1). The frame has pin connections with the foundation. The frame is loaded by four forces. The following relation exists between the plastic moment M_p and the plastic normal force N_p (Fig. 2).

$$N_p = \beta \frac{M_p}{a}$$

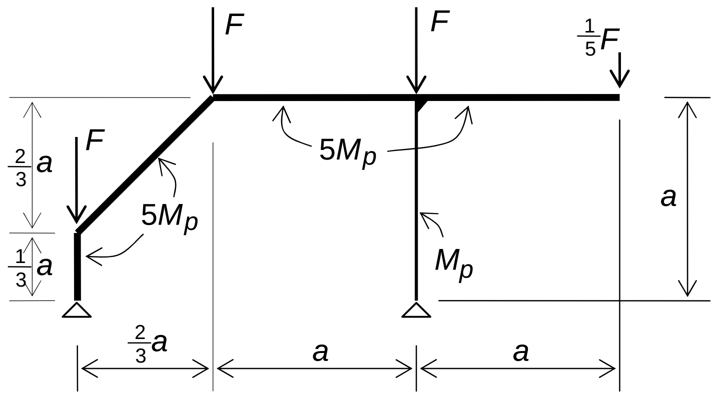


Figure 1. Frame

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

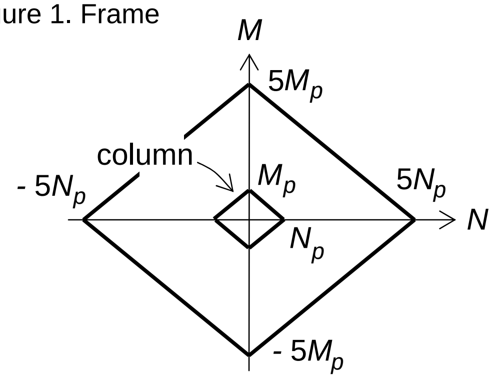


Figure 2. Yield contours of the frame members

a Assume $\beta \to \infty$. Determine the collapse load F for each possible mechanism. Write the collapse loads as functions of M_p and a. What is the decisive collapse load? (1.5 points)

b Assume $\beta \to \infty$. Draw the bending moment diagram of the structure at the moment of collapse. (1 point)

c Assume $\beta = 43\sqrt{2}$. Choose one of the following problems (You need not do both).

– Determine the largest lower-bound for F.

– Determine the smallest upper-bound for F.

For the upper-bound you only need to write down the equations and not to solve them. Use the decisive mechanism of problem **1a**. (2 points)

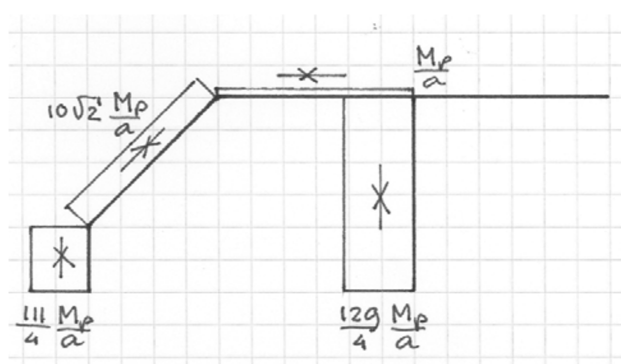
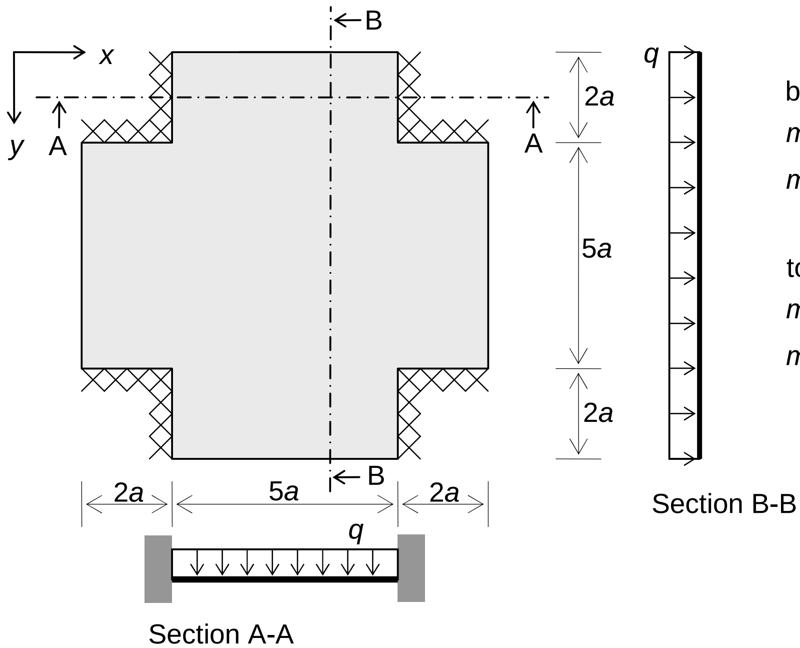


Figure 3. Normal force diagram of the frame

Problem 2

A plate is fixed at eight edges and free at four edges (Figure 3). The plate carries an evenly distributed load q [kN/m²]. The plate is homogeneous.



bottom reinforcement
$m_{px} = 2m_p$
$m_{py} = m_p$

top reinforcement
$m'_{px} = 3m_p$
$m'_{py} = 2m_p$

Figure 4. Plate with fixed and free edges

a We consider the yield line patterns of Figure 5. Which of these patterns give kinematically possible mechanisms. (1 point)

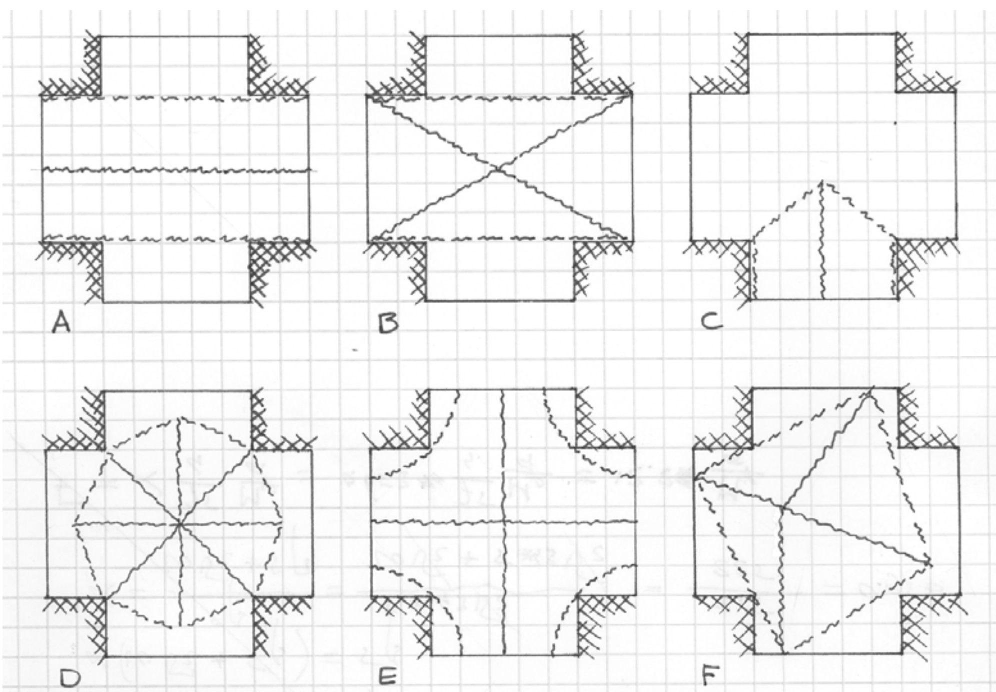


Figure 5. Yield line patterns of problem 2a

b We consider the yield line pattern of Figure 6. Determine an <u>upper bound</u> for q expressed in m_p and a. (1.5 points)

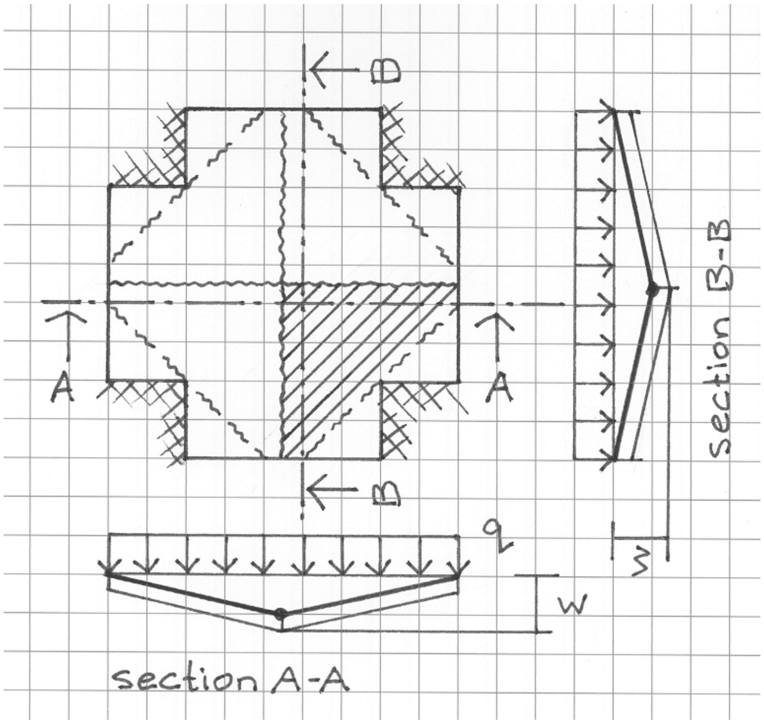


Figure 6. Yield line pattern of problem 2b

c Determine the largest <u>lower-bound</u> for q using torsion free beams ($m_{xy} = 0$) in the x direction and y direction. (1.5 points)

Problem 3

a Consider a small plate part with only a torsion moment $m_{xy} > 0$ (Fig. 7). The reinforcement is in the directions x and y. What is the direction of a positive yield line when $m_{px} > m_{py}$? Choose A, B or C. (0.5 points)

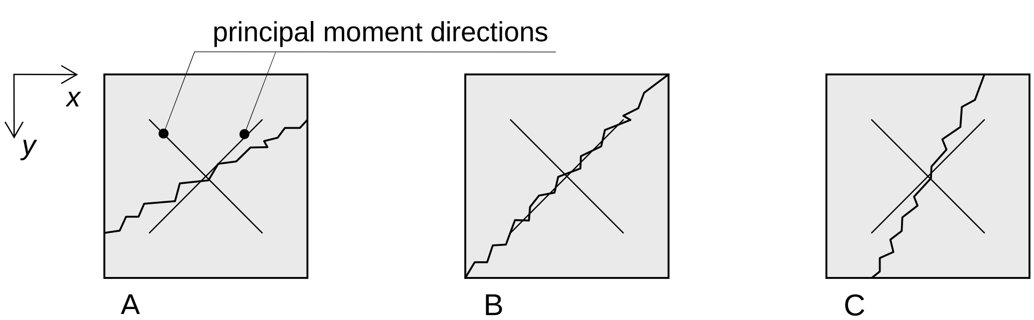
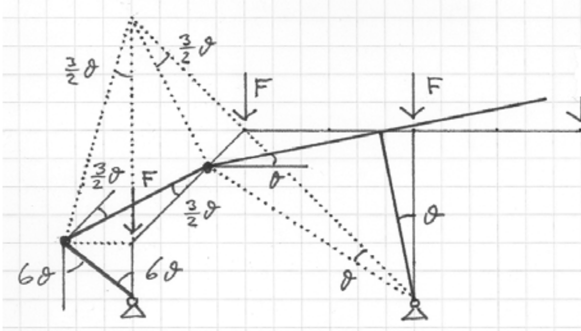


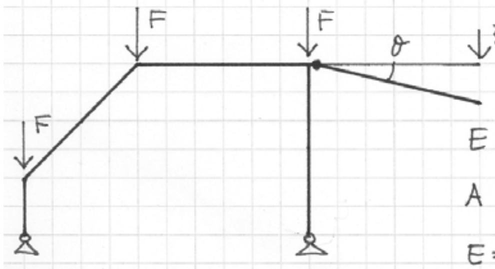
Figure 7. Plate parts with yield lines

b No homogenous isotropic plate can carry a point load larger than $F = 4\pi m_p$. Explain this. (1 point)

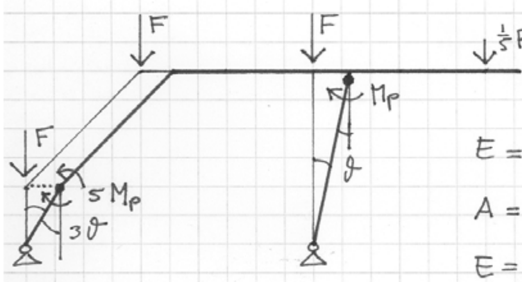
Answer to Problem 1a 18 January 2007



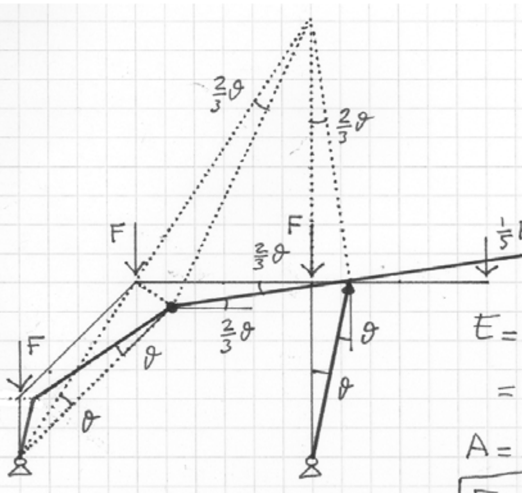
$$E = 5M_p\left(6\vartheta + \tfrac{3}{2}\vartheta\right) + 5M_p\left(\tfrac{3}{2}\vartheta + \vartheta\right)$$
$$= 50\,M_p\vartheta$$
$$A = Fa\vartheta - \tfrac{1}{5}Fa\vartheta = \tfrac{4}{5}aF\vartheta$$
$$E = A \Rightarrow \quad F = \frac{125}{2}\frac{M_p}{a}$$



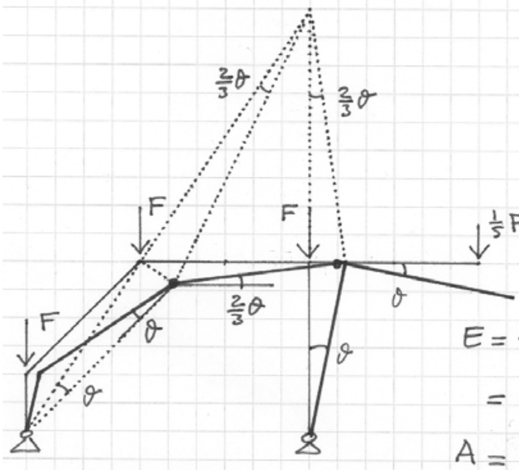
$$E = 5M_p\,\vartheta$$
$$A = \tfrac{1}{5}Fa\vartheta$$
$$E = A \Rightarrow \quad F = 25\frac{M_p}{a}$$



$$E = M_p\,\vartheta + 5M_p\,3\vartheta$$
$$A = F * 0$$
$$E = A \Rightarrow \quad F = \infty$$

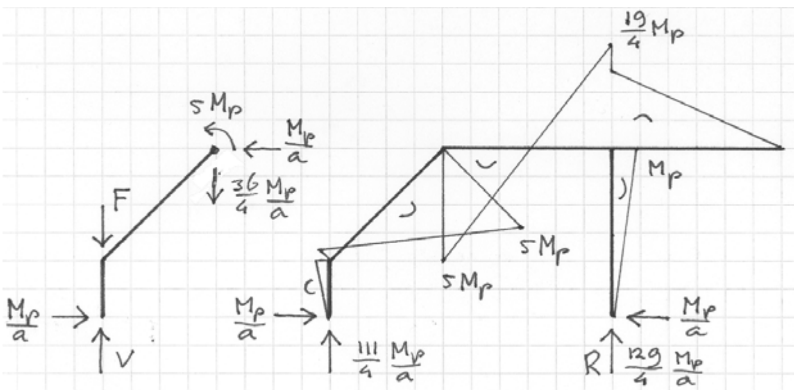


$$E = 5M_p\left(\vartheta + \tfrac{2}{3}\vartheta\right) + M_p\left(\tfrac{2}{3}\vartheta + \vartheta\right)$$
$$= 10\,M_p\vartheta$$
$$A = F\,\tfrac{2}{3}\vartheta a - \tfrac{1}{5}F\,\tfrac{2}{3}\vartheta a = \tfrac{8}{15}F\vartheta a$$
$$\boxed{F = \frac{75}{4}\frac{M_p}{a}} \leftarrow \text{decisive}$$



$$E = 5M_p\left(\vartheta + \tfrac{2}{3}\vartheta\right) + 5M_p\left(\tfrac{2}{3}\vartheta + \vartheta\right)$$
$$= \frac{50}{3} M_p \vartheta$$
$$A = F\,\tfrac{2}{3}\vartheta a + \tfrac{1}{5} F \vartheta a = \frac{13}{15} F\vartheta a$$
$$E = A \;\Rightarrow\; F = \frac{250}{13}\,\frac{M_p}{a}$$

Answer to Problem 1b



$$5M_p + F\,\tfrac{2}{3}a + \frac{M_p}{a}a - V\tfrac{2}{3}a = 0 \;\Rightarrow\; V = \frac{111}{4}\frac{M_p}{a}$$
$$F + F + F + \tfrac{1}{5}F - \frac{111}{4}\frac{M_p}{a} - R = 0 \;\Rightarrow\; R = \frac{129}{4}\frac{M_p}{a}$$

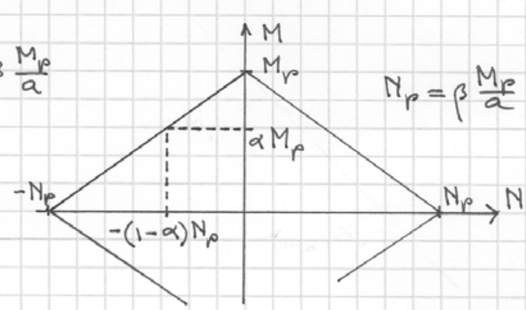
Answer to problem 1c Lower-bound

We reduce the loading, moments and normal forces such that everywhere the yield condition is fulfilled.
In the column head with a moment αM_p we can allow a force $(1-\alpha)N_p$.

$$N = \alpha\,\frac{129}{4}\,\frac{M_p}{a} = (1-\alpha)N_p = (1-\alpha)\beta\,\frac{M_p}{a}$$

$$N_p = \beta\,\frac{M_p}{a}$$

$$\alpha\,\frac{129}{4} = (1-\alpha)\beta$$
$$\alpha\left(\frac{129}{4} + \beta\right) = \beta$$
$$\alpha = \frac{\beta}{\frac{129}{4} + \beta} = \frac{43\sqrt{2}}{\frac{129}{4} + 43\sqrt{2}} = 0{,}653$$



In the inclined beam with a moment $\alpha 5 M_p$ we can allow a force $(1-\alpha) 5 N_p$

$$N = \alpha\, 10\sqrt{2}\, \frac{M_p}{a} = (1-\alpha)\, 5 N_p = (1-\alpha)\, 5\beta \frac{M_p}{a}$$

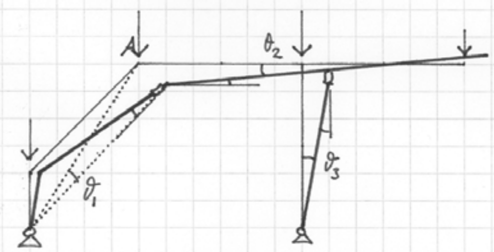
$$\alpha\, 10\sqrt{2} = (1-\alpha)\, 5\beta$$

$$\alpha\,(10\sqrt{2} + 5\beta) = 5\beta$$

$$\alpha = \frac{5\beta}{10\sqrt{2} + 5\beta} = \frac{5 \times 43\sqrt{2}}{10\sqrt{2} + 5 \times 43\sqrt{2}} = \frac{215}{225} = 0{,}956$$

$$F = \alpha \frac{75}{4} \frac{M_p}{a} = 0{,}653 \frac{75}{4} \frac{M_p}{a} = 12{,}2 \frac{M_p}{a}$$

Answer to Problem 1c Upper-bound



$$u = \vartheta \frac{a}{\beta} \qquad \beta = 43\sqrt{2}$$

horizontal displacement of A →

$$\vartheta_1 a - (\vartheta_1 + \vartheta_2) \frac{a}{\beta} \frac{1}{\sqrt{2}} = \vartheta_3 a$$

vertical displacement of A ↓

$$\vartheta_1 \frac{2}{3} a + (\vartheta_1 + \vartheta_2) \frac{a}{\beta} \frac{1}{\sqrt{2}} = (\vartheta_2 + \vartheta_3) \frac{a}{\beta} + \vartheta_2 a$$

$$A = F \left[\vartheta_1 \frac{2}{3} a + (\vartheta_1 + \vartheta_2) \frac{a}{\beta} \frac{1}{\sqrt{2}} \right] + F (\vartheta_2 + \vartheta_3) \frac{a}{\beta} - \frac{1}{5} F \left[\vartheta_2 a - (\vartheta_2 + \vartheta_3) \frac{a}{\beta} \right]$$

$$E = 5 M_p (\vartheta_1 + \vartheta_2) + M_p (\vartheta_2 + \vartheta_3)$$

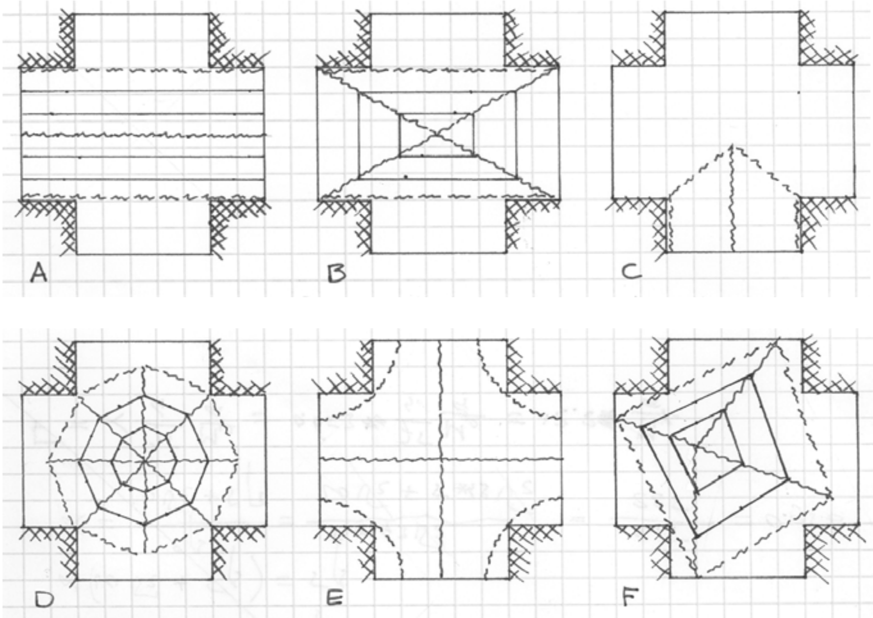
$$A = E$$

the solution is computed by Maple

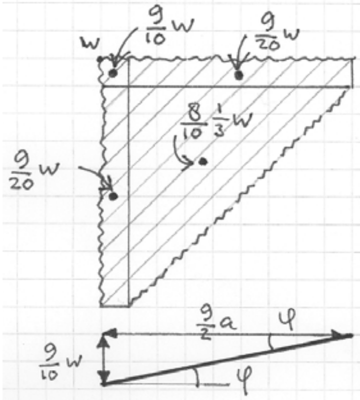
$$\vartheta_1 = 1.52\, \vartheta_2 \qquad \vartheta_3 = 1.49\, \vartheta_2 \qquad F = 16.9 \frac{M_p}{a}$$

Answer to Problem 2a

Kinematically possible are patterns A, B, D and F. The figure below shows the altitude lines of the deformed mechanisms.



Answer to Problem 2b



$$A = 4\left[q\frac{(4a)^2}{2}\frac{8}{30}w + 2\left(q\,4a\,\tfrac{1}{2}a\,\tfrac{9}{20}w\right) + q\left(\tfrac{a}{2}\right)^2\tfrac{9}{10}w \right]$$

$$= \frac{499}{30}\, q a^2 w$$

$$\varphi = \frac{\frac{9}{10}w}{\frac{9}{2}a} = \frac{1}{5}\frac{w}{a}$$

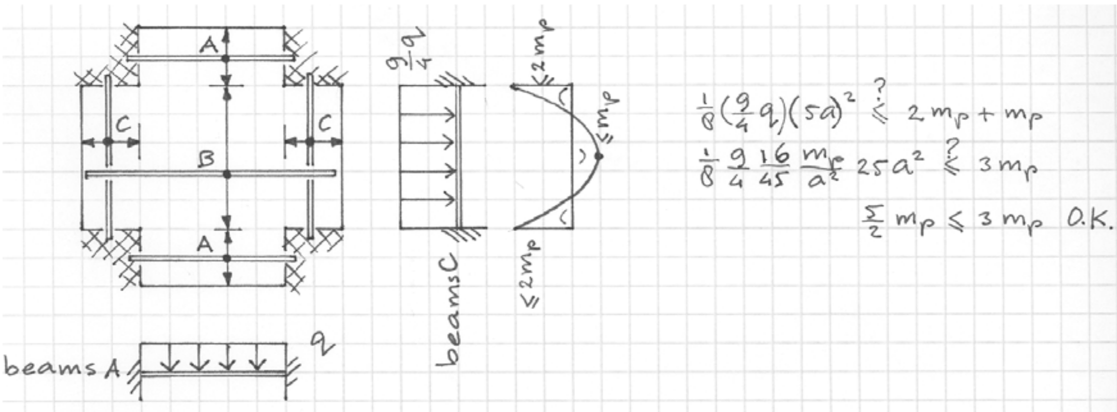
$$E = 2m_p\,2\,\tfrac{1}{5}\tfrac{w}{a}\,9a + m_p\,2\,\tfrac{1}{5}\tfrac{w}{a}\,9a +$$

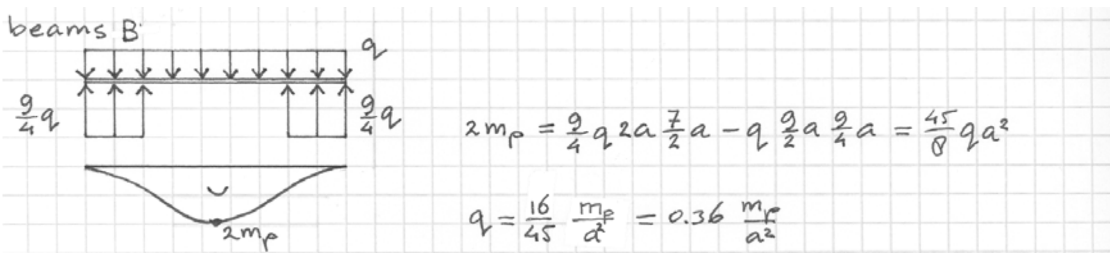
$$+\,4\left[3m_p\,\tfrac{1}{5}\tfrac{w}{a}\,4a + 2m_p\,\tfrac{1}{5}\tfrac{w}{a}\,4a\right]$$

$$= \frac{134}{5}\, m_p w$$

$$A = E \Rightarrow \quad q = \frac{804}{499}\,\frac{m_p}{a^2} = 1{,}61\,\frac{m_p}{a^2}$$

Answer to Problem 2c





Answer to Problem 3a

A: The yield line will try crossing mainly weaker reinforcing bars (Lecture book p. 64)

Answer to Problem 3b

$4\pi m_p$ is the collapse load of a circular plate fixed on the edge. This circular yield line pattern (Lecture book Fig. 10.2) is possible in any plate for a point load somewhere on its interior. If the point load acts on an edge the upper-bound is smaller (Lecture book Fig. 10.4). Therefore, $4\pi m_p$ is a general upper-bound.