Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top right-hand of your work.

Exam CT4150 Plastic Analysis of Structures
Wednesday 20 June 2007, 9:00 – 12:00 hours

Problem 1

A frame consists of two columns with strengths M_p and one beam with strength $6M_p$ (Fig. 1). The left-hand beam has a fixed support while the right-hand beam has a roller support, The frame is loaded by a vertical force $30F$ and a horizontal force F. The following relation exists between the plastic moment M_p and the plastic normal force N_p (Fig. 2).

$$N_p = \beta \frac{M_p}{a}$$

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

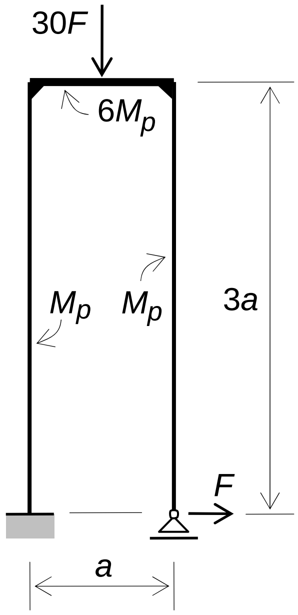

Figure 1. Frame

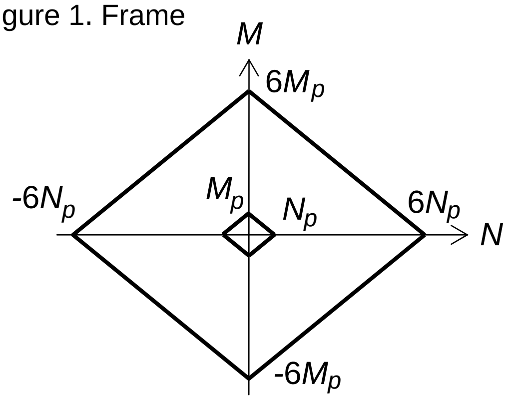

Figure 2. Yield contour

a Assume $\beta \to \infty$. Determine the collapse load F for all possible mechanisms. Write the collapse loads as functions of M_p and a. What is the decisive collapse load? (1.5 points)

b Assume $\beta \to \infty$. Draw the bending moment diagram and normal force diagram for the structure at the moment of collapse. (1 point)

c Assume $\beta = 60$. Choose one of the following problems (You need not do both).

– Determine the largest lower-bound for F.

– Determine the smallest upper-bound for F.

Use the mechanism of Figure 3. (2 points)

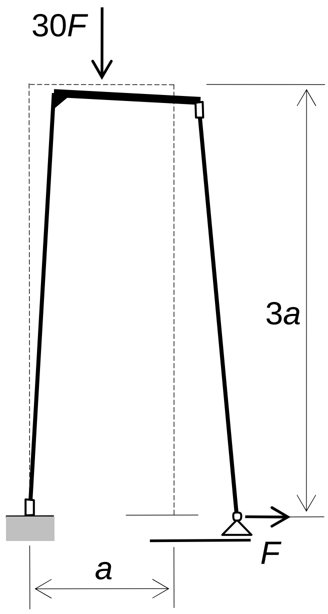

Figure 3. Mechanism

Problem 2

A plate is fixed at three edges and free at one edge (Fig. 4). The plate carries a hydrostatically distributed load $q\left(1-\frac{y}{a}\right)$ [kN/m²]. The plate is homogeneous. The reinforcement directions are x and y. $m_{px} = 3m_p$ and $m'_{px} = m'_{py} = m_{py} = m_p$.

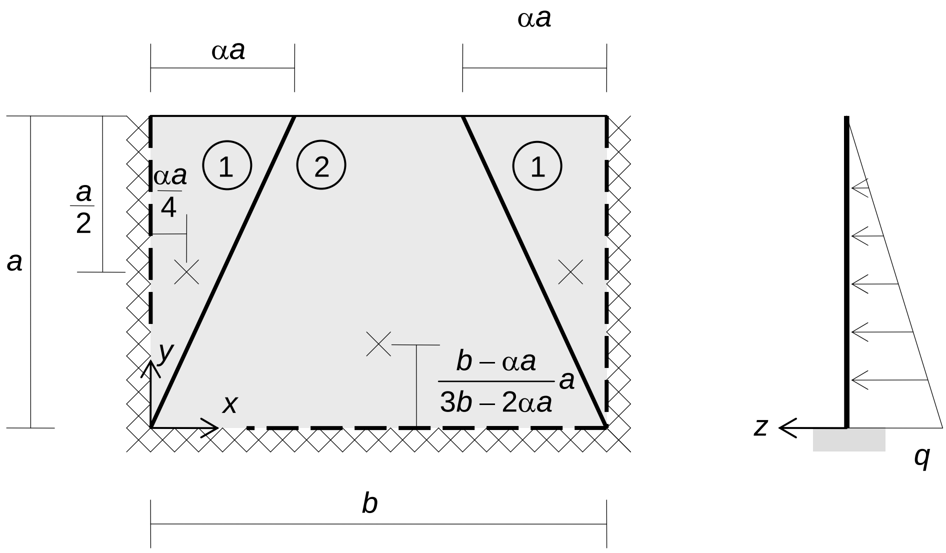

Figure 4. Hydrostatically loaded plate

a Consider the yield line patterns of Figure 5. Which of these patterns give kinematically possible mechanisms. (1 point)

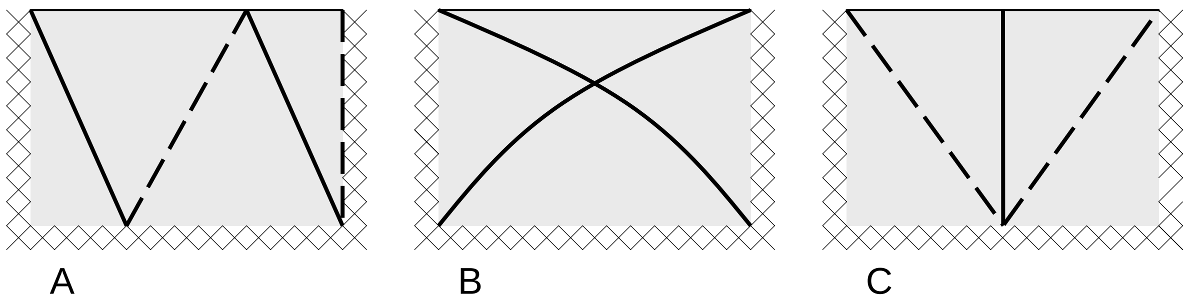

Figure 5. Yield line patterns of problem 2a

b Consider the yield line pattern of Figure 4. Determine an <u>upper bound</u> for q expressed in m_p and a, b and α. The crosses in Figure 4 show the positions of the resulting forces on the plate parts. The values of the resulting forces on the plate parts are $F_1 = \frac{1}{6}\alpha q a^2$ and $F_2 = \frac{1}{6}(3b - 2\alpha a)qa$. (1.5 points)

c Determine the largest <u>lower-bound</u> for q using torsion free beams ($m_{xy} = 0$) in the x direction and y direction. (1.5 points)

Problem 3

a Foundation settlements can be ignored when we calculate the plastic collapse load of a structure. How can this be explained correctly? (0.5 point)

A – Settlement stresses are very small compared to the stresses due to self weight.
B – The plastic hinges in the ultimate limit state make the structure statically determinate.
C – When all foundation piles settle the same amount there will be no redistribution of the force flow.
D – Foundation settlements cannot be ignored. It can cause damage which can become worse due to rain and wind. Expensive repairs can be required, the structure might be closed temporarily reducing its serviceability and in time the structure might even collapse.

b Consider the three plates of Figure 6. Each of them is homogeneous and isotropic with a plastic moment m_p. Each plate is loaded by a perpendicular point load F at the location of the dot. Which of the plates has the largest ultimate load? (1 point)

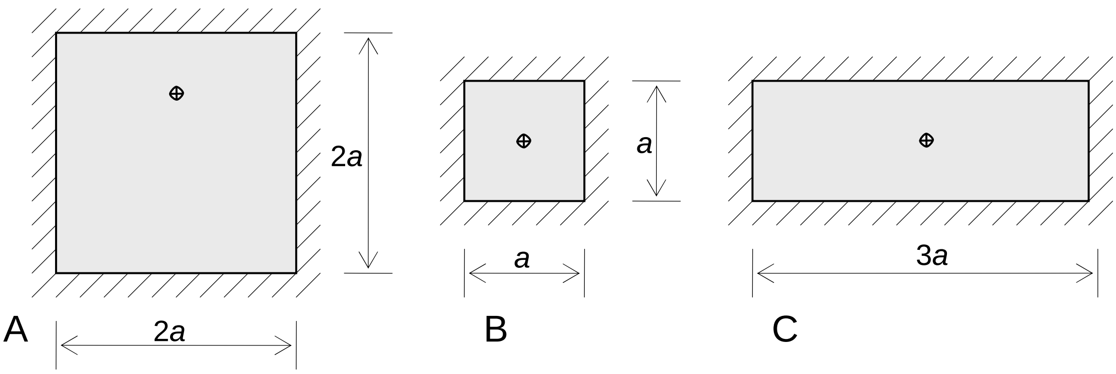

Figure 6. Plates loaded by a point load

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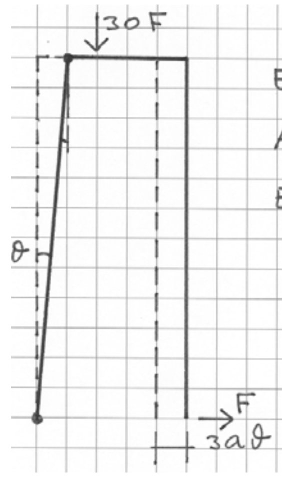
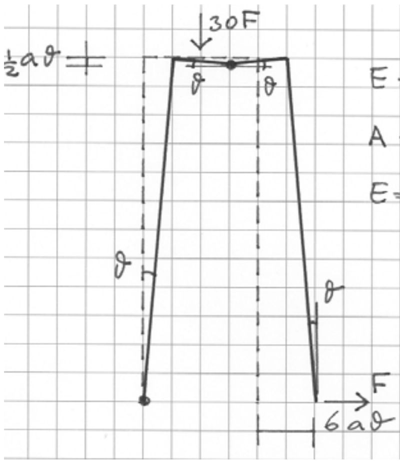
Answer to Problem 1a



$$E = M_p \vartheta + M_p \vartheta = 2 M_p \vartheta$$

$$A = F \vartheta 3a$$

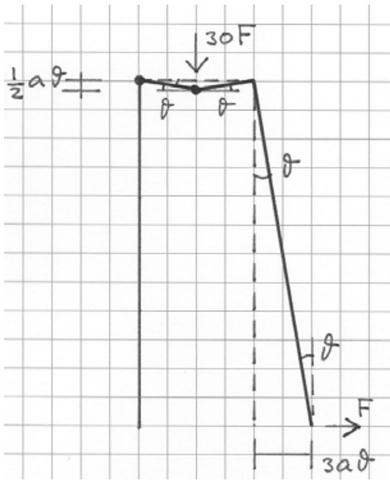
$$E = A \Rightarrow F = \frac{2}{3} \frac{M_p}{a}$$



$$E = M_p \vartheta + 6 M_p (\vartheta + \vartheta) = 13 M_p \vartheta$$

$$A = F\, 6a\vartheta + 30 F \tfrac{1}{2} a \vartheta = 21 F a \vartheta$$

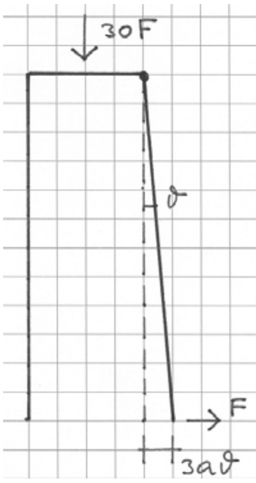
$$E = A \Rightarrow F = \frac{13}{21} \frac{M_p}{a}$$



$$E = 6 M_p (\vartheta + \vartheta) + M_p \vartheta = 13 M_p \vartheta$$

$$A = F\, 3a\vartheta + 30 F \tfrac{1}{2} a \vartheta = 18 F a \vartheta$$

$$E = A \Rightarrow F = \frac{13}{18} \frac{M_p}{a}$$



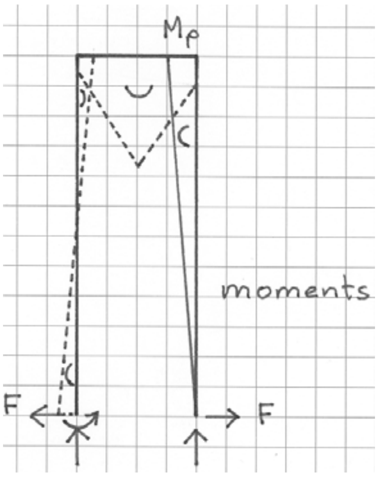
$$E = M_p \vartheta$$

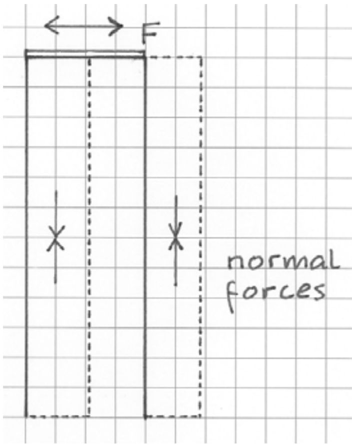
$$A = F\, 3a \vartheta$$

$$E = A \Rightarrow \boxed{F = \frac{1}{3}\frac{M_p}{a}} \text{ decisive}$$

Answer to Problem 1b

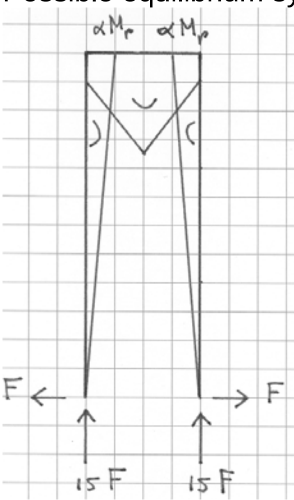
The decisive mechanism is a partial mechanism. Therefore, the force distribution is only partially determined. We estimate (dashed lines) the rest of the moment distribution and normal force distribution.





Answer to problem 1c Lower-bound

Possible equilibrium system



$$F\, 3a = \alpha M_p \Rightarrow F = \alpha \frac{1}{3}\frac{M_p}{a}$$

$$15F = (1-\alpha) N_p \Rightarrow$$

$$15\alpha \frac{1}{3}\frac{M_p}{a} = (1-\alpha) N_p = (1-\alpha)\beta \frac{M_p}{a}$$

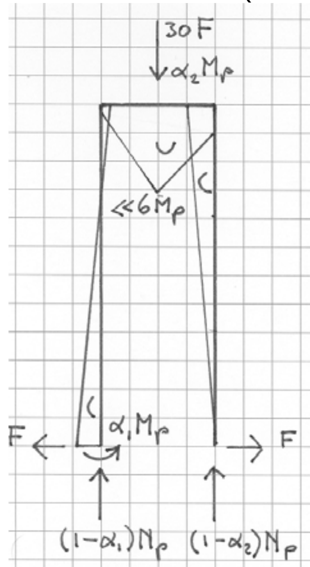
$$5\alpha = (1-\alpha)\beta$$

$$5\alpha + \beta\alpha = \beta$$

$$\alpha = \frac{\beta}{5+\beta} = \frac{60}{65} = \frac{12}{13}$$

$$F = \frac{12}{13}\frac{1}{3}\frac{M_p}{a} = \frac{4}{13}\frac{M_p}{a} = 0{,}31 \frac{M_p}{a}$$

Alternative answer (lower-bound)



$N_p = \beta \frac{M_p}{a}$ $\quad \beta = 60$

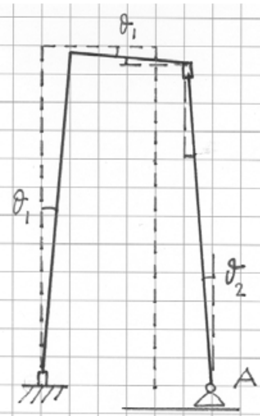
equilibrium equations

$$\begin{cases} F\,3a = \alpha_2 M_p \\ \alpha_1 M_p = 30F\,\frac{1}{2}a - (1-\alpha_2)N_p a \\ 30F = (1-\alpha_1)N_p + (1-\alpha_2)N_p \end{cases}$$

solution

$$\begin{cases} \alpha_1 = \frac{360}{397} \\ \alpha_2 = \frac{372}{397} \\ F = \frac{124}{397}\frac{M_p}{a} \end{cases}$$

Answer to Problem 1c Upper-bound



$u = \vartheta \frac{a}{\beta}$

$\beta = 60$

displacement of point A ↓

$$\vartheta_1 \frac{a}{\beta} + \vartheta_1 a - (\vartheta_1 + \vartheta_2)\frac{a}{\beta} = 0 \quad \Rightarrow \quad \vartheta_1 = \frac{1}{\beta}\vartheta_2$$

$$E = M_p \vartheta_1 + M_p(\vartheta_1 + \vartheta_2) = M_p\left(1 + \frac{2}{\beta}\right)\vartheta_2$$

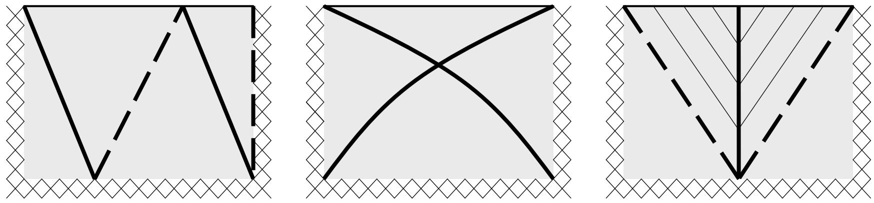
$$A = F\left(\vartheta_1 3a + \vartheta_2 3a\right) + 30F\left(\vartheta_1 \frac{a}{\beta} + \vartheta_1 \frac{1}{2}a\right)$$

$$= Fa\left(3 + \frac{18}{\beta} + \frac{30}{\beta^2}\right)\vartheta_2$$

$$E = A \quad \Rightarrow \quad F = \frac{1 + \frac{2}{\beta}}{3 + \frac{18}{\beta} + \frac{30}{\beta^2}}\frac{M_p}{a} = \frac{124}{397}\frac{M_p}{a} = 0.31\frac{M_p}{a}$$

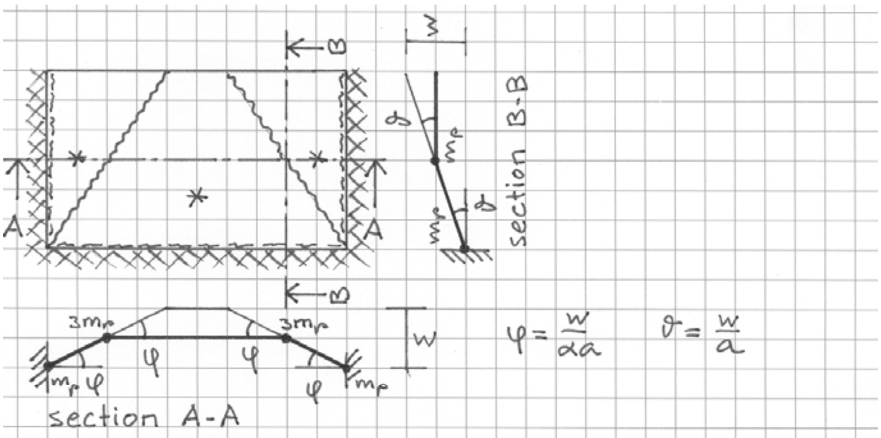
Answer to Problem 2a

Kinematically possible is pattern C. The figure below shows the altitude lines of the deformed mechanism.



A B C

Answer to Problem 2b



$$\varphi = \frac{w}{\alpha a} \qquad \vartheta = \frac{w}{a}$$

$$E = (3 m_p \varphi a + m_p \vartheta \alpha a)\, 2 + (m_p \varphi a)\, 2 + m_p \vartheta b$$

$$= m_p w \left(\frac{8}{\alpha} + 2\alpha + \frac{b}{a} \right)$$

$$A = \left(F_1 \frac{w}{4}\right) 2 + F_2 \frac{b - \alpha a}{3b - 2\alpha a} w$$

$$= \frac{1}{12} w q a^2 \left(2 \frac{b}{a} - \alpha \right)$$

$$E = A \Rightarrow \quad q = 12 \frac{2\alpha^2 a + 8a + b\alpha}{\alpha(\alpha a - 2b)} \frac{m_p}{a^2} \qquad 2\alpha a \leq b$$

<u>Encore</u> (not an exam question)

The plate resultants and there positions have been computed by Maple

$$F_1 = \int_0^a \int_0^{\alpha y} q \left(1 - \frac{y}{a} \right) dx\, dy = \frac{1}{6} \alpha q a^2$$

$$x_1 = \frac{\int_0^a \int_0^{\alpha y} x q \left(1 - \frac{y}{a} \right) dx\, dy}{F_1} = \frac{1}{4} \alpha a$$

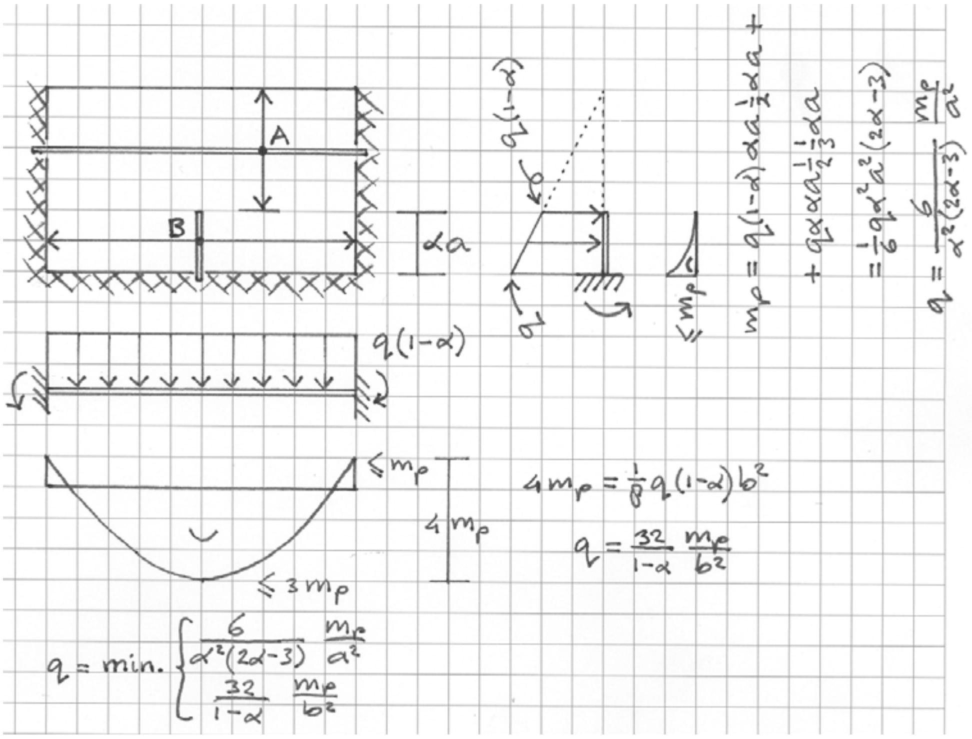
$$y_1 = \frac{\int_0^a \int_0^{\alpha y} y q \left(1 - \frac{y}{a} \right) dx\, dy}{F_1} = \frac{1}{2} a$$

$$F_2 = \int_0^a \int_{\alpha y}^{b-\alpha y} q\left(1-\frac{y}{a}\right) dx\, dy = \tfrac{1}{6}(3b - 2\alpha a) qa$$

$$x_2 = \frac{\int_0^a \int_{\alpha y}^{b-\alpha y} x\, q\left(1-\frac{y}{a}\right) dx\, dy}{F_2} = \tfrac{1}{2} b$$

$$y_2 = \frac{\int_0^a \int_{\alpha y}^{b-\alpha y} y\, q\left(1-\frac{y}{a}\right) dx\, dy}{F_2} = \frac{b - \alpha a}{3b - 2\alpha a} a$$

Answer to Problem 2c



Answer to Problem 3a

B is correct. Statement A is wrong. Statement C is true but not an answer to the problem. Statement D is true; a foundation settlement may be a first step in a chain of events that leads to collapse, however, in a ductile structure it is never the cause of immediate collapse.

Answer to Problem 3b

$$F_A = 2\left(2\pi - \frac{\pi}{2} + 1\right) m_p = 11.4\, m_p$$

$F_B = 8\, m_p$ (plate lecture book page 46 - 48)

$$F_C = (4 + 2\pi) m_p = 10.3\, m_p$$

Therefore, plate A has the largest collapse load.
Note that the plates would have had identical collapse loads if the edges were fixed.