Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Exam CIE4150 Plastic Analysis of Structures
Thursday 11 April 2019, 13:30 – 16:30 hours

Write your <u>name</u> and <u>study number</u> at the top of your work.

Also write whether you were a <u>member</u> of the elastic team, plastic team or no team.

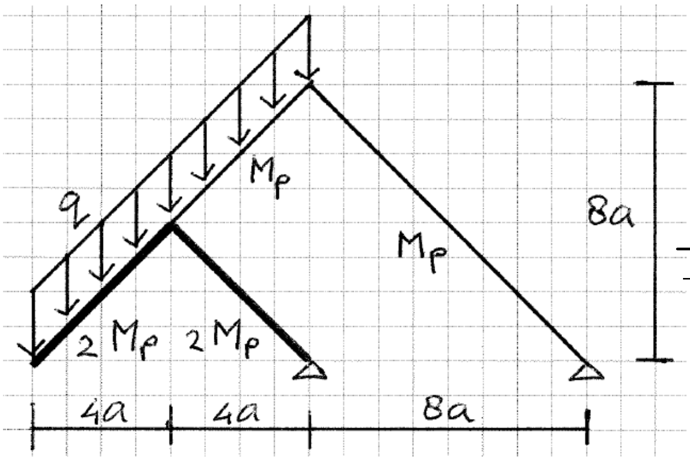


Figure 1. Frame structure

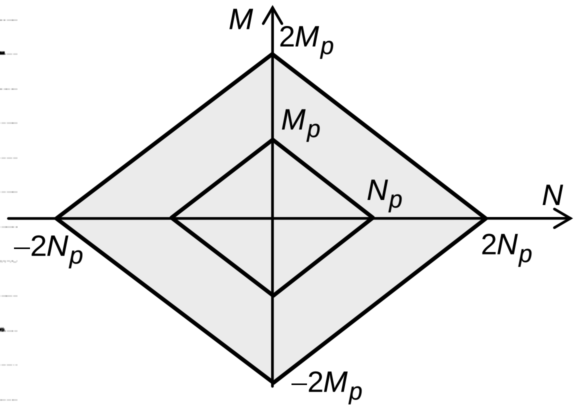


Figure 2. Yield contours

Problem 1

A frame consists of two columns, a beam and a cantilever (Fig.1). The right hand column and the beam have a strength M_p . The left hand beam and the cantilever have a strength $2M_p$. The elements are rigidly connected. The support consist of two hinges. The structure is loaded by an evenly distributed line load *q* per length of beam (self-weight). The relation of Figure 2 exists between the plastic moments and the plastic normal forces.

$$N_p = \beta \frac{M_p}{a}$$

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

a Assume $\beta \to \infty$. Determine the collapse load *q* for all possible mechanisms. Write the collapse loads as functions of M_p and *a*. What is the decisive collapse load? (1.5 point)

b Assume $\beta \to \infty$. Draw the bending moment diagram and normal force diagram for the structure at the moment of collapse. (1.5 points)

c Assume $\beta = 16$. Choose one of the following problems (You need not do both).
– Determine the largest <u>lower-bound</u> for *q*.
– Determine the smallest <u>upper-bound</u> for *q*.
You only need to write down the equations and not solve the equations (1.5 points).

Problem 2

A reinforced concrete plate has simply supported edges and free edges (Fig. 3). It carries an evenly distributed load p [kN/m^2]. There is no other load on the plate. The plate is homogeneous and orthotropic.

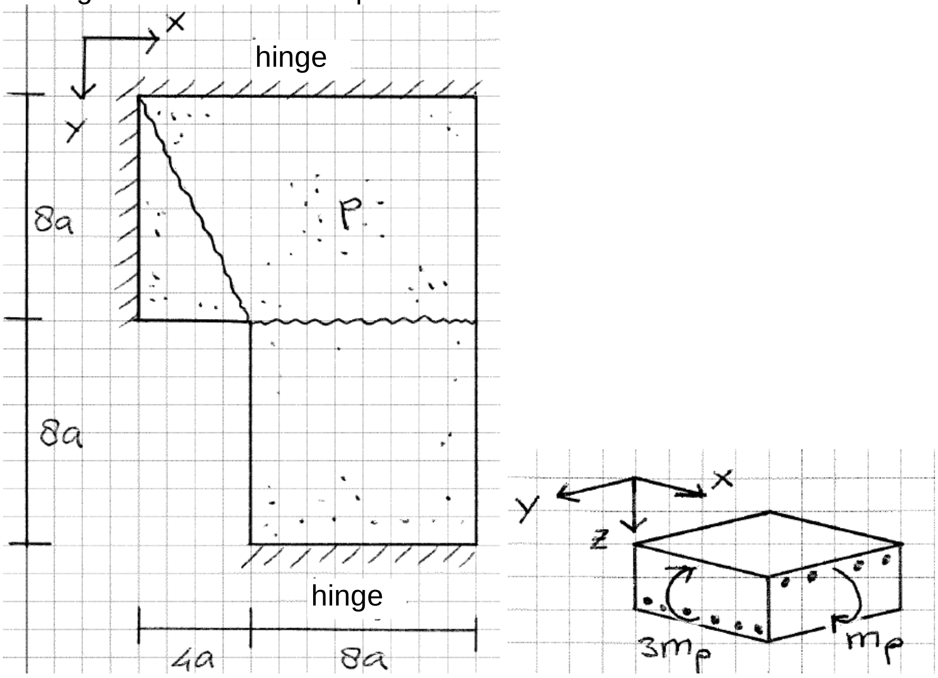


Figure 3. Plate dimensions and reinforcement

a Consider the yield line patterns of Figure 4. Which of these patterns give kinematically possible mechanisms? (1 point)

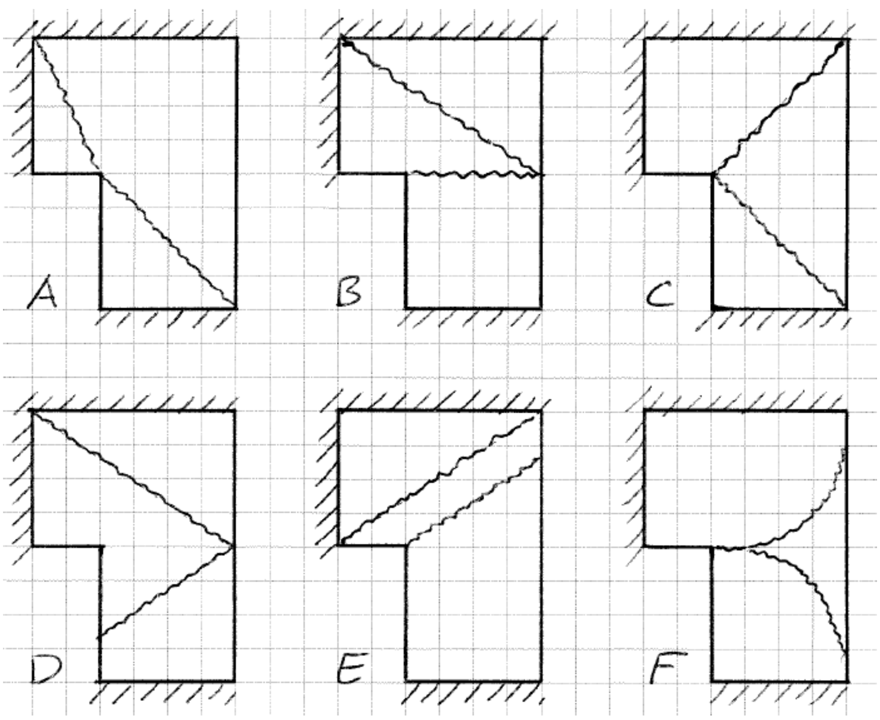


Figure 4. Yield line patterns of problem 2a

b Consider the yield line pattern of Figure 3. Determine an <u>upper bound</u> for p expressed in m_p and a (1.5 point).

c Determine the largest <u>lower-bound</u> for p using torsion free beams ($m_{xy} = 0$). You only need to write down the equations and not solve the equations. (1.5 point)

Problem 3

a At what shear stress τ do metals yield? Choose A, B, C or D. (0.5 points) (f_y is the yield strength.)

A $\tau = \pm f_y$

B $\tau = \pm \frac{f_y^2}{\sqrt{3}}$

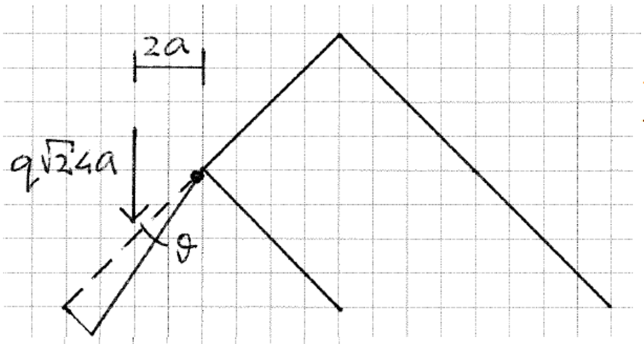
C $\tau = \pm 2 \frac{f_y}{\sqrt{3}}$

D $\tau = \pm \frac{f_y}{\sqrt{3}}$

b The rhombic yield contour (problem 1c) has a property that other yield contours do not have. Which property is this? Choose A, B, C or D. (0.5 points)

A Virtual work: The internal virtual work can be calculated by $M\vartheta + Nu = M_p\vartheta$.
B Normality: The deformation is described by the normal vector; except for in corners.
C Convexity: The yield contour is convex.
D Accuracy: The yield contour is accurate for most steel beam symmetrical cross-sections.

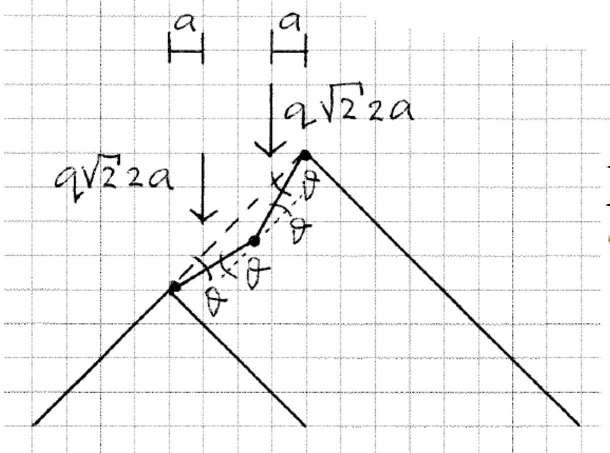
c Which yield contour is suitable for modelling reinforced concrete plates?
How do we know this is true? (0.5 points)



$E := 2\,Mp\cdot t:$
$A := q\cdot \mathrm{sqrt}(2)\cdot 4\,a\cdot 2\,a\cdot t:$
$q := solve(E = A, q);\ evalf(q);$

$$q := \frac{1}{8}\,\frac{Mp\sqrt{2}}{a^2}$$

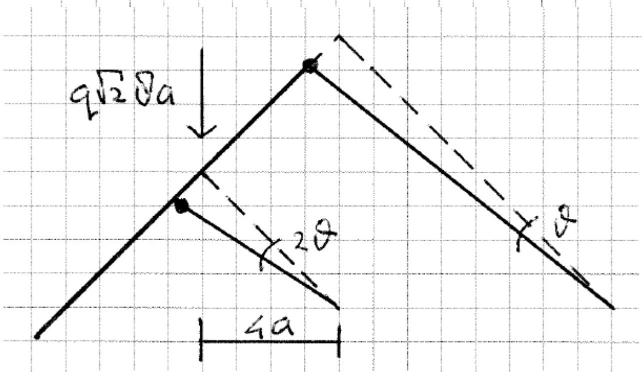
$$\frac{0.1767766952\,Mp}{a^2}$$



$E := Mp\cdot t + Mp\cdot (t + t) + Mp\cdot t:$
$A := q\cdot \mathrm{sqrt}(2)\cdot 2\,a\cdot a\cdot t + q\cdot \mathrm{sqrt}(2)\cdot 2\,a\cdot a\cdot t:$
$q := solve(E = A, q);\ evalf(q);$

$$q := \frac{1}{2}\,\frac{Mp\sqrt{2}}{a^2}$$

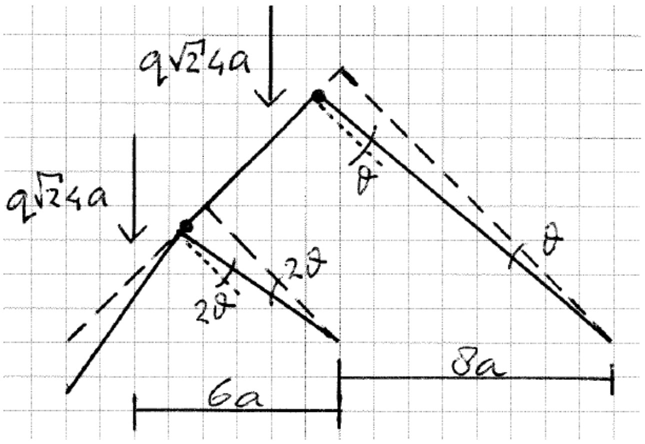
$$\frac{0.7071067810\,Mp}{a^2}$$



$E := 2\,Mp\cdot 2\,t + Mp\cdot t:$
$A := q\cdot \mathrm{sqrt}(2)\cdot 8\,a\cdot 4\,a\cdot 2\,t:$
$q := solve(E = A, q);\ evalf(q);$

$$q := \frac{5}{128}\,\frac{Mp\sqrt{2}}{a^2}$$

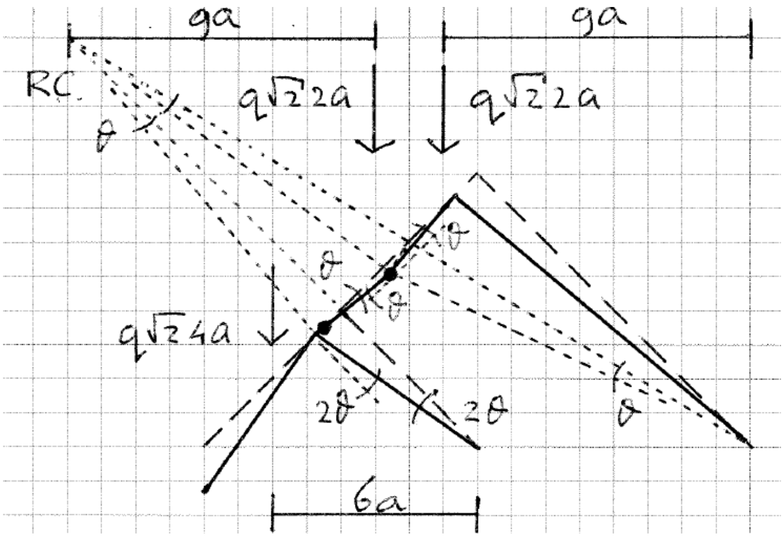
$$\frac{0.05524271727\,Mp}{a^2}$$



$E := Mp\cdot 2\,t + Mp\cdot t:$
$A := q\cdot \mathrm{sqrt}(2)\cdot 4\,a\cdot 6\,a\cdot 2\,t + q\cdot \mathrm{sqrt}(2)\cdot 4\,a\cdot 8\,a\cdot t:$
$q := solve(E = A, q);\ evalf(q);$

decisive $$\boxed{q := \frac{3}{160}\,\frac{Mp\sqrt{2}}{a^2}}$$

$$\frac{0.02651650429\,Mp}{a^2}$$



$E := Mp \cdot (2\,t + t) + Mp \cdot (t + t)$:

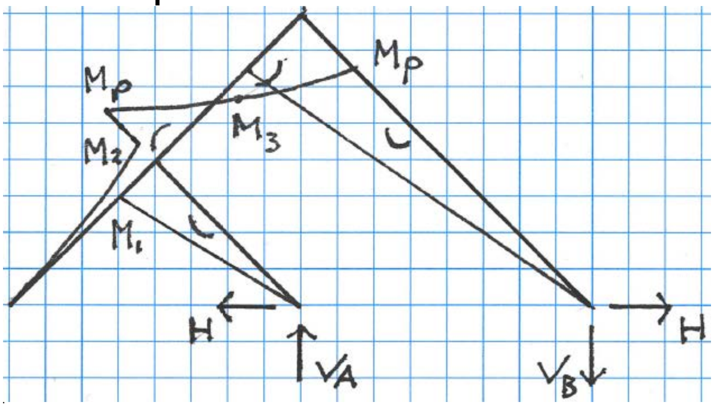
$A := q \cdot \mathrm{sqrt}(2) \cdot 4\,a \cdot 6\,a \cdot 2\,t + q \cdot \mathrm{sqrt}(2) \cdot 2\,a \cdot 9\,a \cdot t + q \cdot \mathrm{sqrt}(2) \cdot 2\,a \cdot 9\,a \cdot t$:

$q := solve(E = A, q);\ evalf(q);$

$$q := \frac{5}{168}\,\frac{Mp\sqrt{2}}{a^2}$$

$$\frac{0.04208968934\,Mp}{a^2}$$

Answer to problem 1b



> $q := \frac{3}{160} \cdot \mathrm{sqrt}(2) \cdot \frac{Mp}{a^2}$:

> $VA := solve(q \cdot \mathrm{sqrt}(2) \cdot 8 \cdot a \cdot 12 \cdot a - VA \cdot 8 \cdot a = 0, VA);$

$$VA := \frac{9}{20}\,\frac{Mp}{a}$$

> $VB := solve(q \cdot \mathrm{sqrt}(2) \cdot 8 \cdot a - VA + VB = 0, VB);$

$$VB := \frac{3}{20}\,\frac{Mp}{a}$$

> $H := solve(Mp = H \cdot 8 \cdot a - VB \cdot 8 \cdot a, H);$

$$H := \frac{11}{40}\,\frac{Mp}{a}$$

> $M1 := VA \cdot 4 \cdot a - H \cdot 4 \cdot a;$

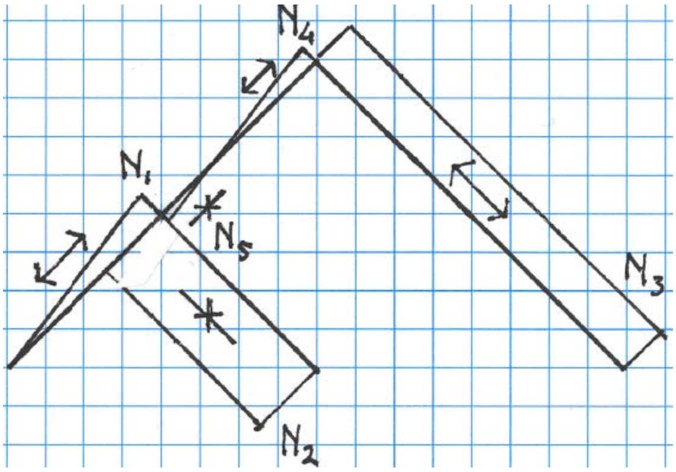
$$M1 := \frac{7}{10}\,Mp$$

> $M2 := q \cdot \text{sqrt}(2) \cdot 4 \cdot a \cdot 2 \cdot a;$

$$M2 := \frac{3}{10}\, Mp$$

> $M3 := H \cdot 6 \cdot a - VB \cdot 10 \cdot a - q \cdot \text{sqrt}(2) \cdot 2 \cdot a \cdot a;$

$$M3 := \frac{3}{40}\, Mp$$



> $N1 := \dfrac{q \cdot \text{sqrt}(2) \cdot 4 \cdot a}{\text{sqrt}(2)};$

$$N1 := \frac{3}{40}\, \frac{\sqrt{2}\, Mp}{a}$$

> $N2 := -\dfrac{VA}{\text{sqrt}(2)} - \dfrac{H}{\text{sqrt}(2)};$

$$N2 := -\frac{29}{80}\, \frac{\sqrt{2}\, Mp}{a}$$

> $N3 := \dfrac{VB}{\text{sqrt}(2)} + \dfrac{H}{\text{sqrt}(2)};$

$$N3 := \frac{17}{80}\, \frac{\sqrt{2}\, Mp}{a}$$

> $N4 := \dfrac{H}{\text{sqrt}(2)} - \dfrac{VB}{\text{sqrt}(2)};$

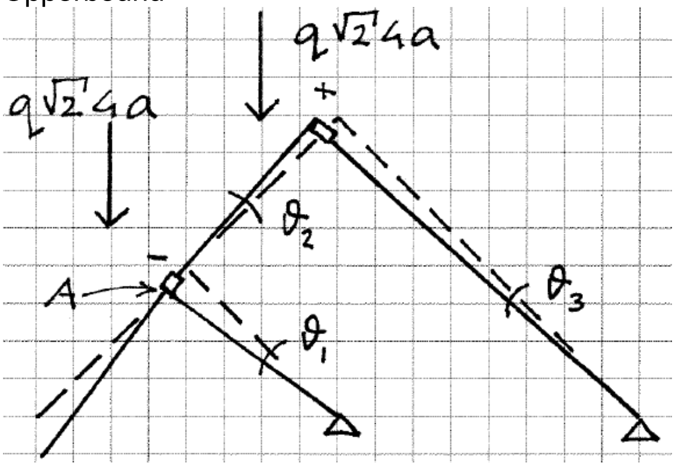
$$N4 := \frac{1}{16}\, \frac{\sqrt{2}\, Mp}{a}$$

> $N5 := N1 - \dfrac{VA}{\text{sqrt}(2)} + \dfrac{H}{\text{sqrt}(2)};$

$$N5 := -\frac{1}{80}\, \frac{\sqrt{2}\, Mp}{a}$$

Answer to problem 1c

Upperbound



> $b := 16.$:

> *# horizontal displacement of A (to the left is positive)*

> $$eq1 := t1 \cdot 4 \cdot a = t3 \cdot 8 \cdot a + \frac{a}{b} \cdot (t3 - t2) \cdot \frac{1}{\operatorname{sqrt}(2)} - t2 \cdot 4 \cdot a - \frac{a}{b} \cdot (t1 - t2) \cdot \frac{1}{\operatorname{sqrt}(2)} :$$

> *# vertical displacement of A (down is positive)*

> $$eq2 := t1 \cdot 4 \cdot a = t3 \cdot 8 \cdot a - \frac{a}{b} \cdot (t3 - t2) \cdot \frac{1}{\operatorname{sqrt}(2)} + t2 \cdot 4 \cdot a - \frac{a}{b} \cdot (t1 - t2) \cdot \frac{1}{\operatorname{sqrt}(2)} :$$

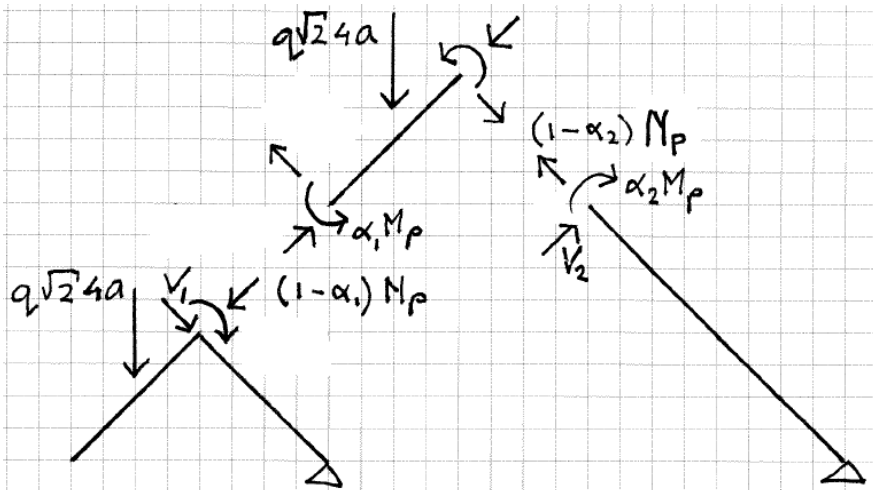
> $$E := Mp \cdot (t1 - t2) + Mp \cdot (t3 - t2) :$$

> $$A := q \cdot \operatorname{sqrt}(2) \cdot 4 \cdot a \cdot 6 \cdot a \cdot t1 + q \cdot \operatorname{sqrt}(2) \cdot 4 \cdot a \cdot \left(t3 \cdot 8 \cdot a - \frac{a}{b} \cdot (t3 - t2) \cdot \frac{1}{\operatorname{sqrt}(2)} + t2 \cdot 2 \cdot a \right) :$$

> $solve(\{eq1, eq2, E = A\}, \{q, t1, t2\});$

$$\left\{ q = \frac{0.02633168343\, Mp}{a^2},\ t1 = 1.978263803\, t3,\ t2 = 0.01092780710\, t3 \right\}$$

Lowerbound



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> b := 16 :
> Np := b/a·Mp :
> F := q·sqrt(2.)·4·a :
> eq1 := F·6·a + (1 − a1)·Np·4·a·sqrt(2) − a1·Mp = 0 :
> eq2 := a1·Mp + a2·Mp − F·2·a − (1 − a2)·Np·4·a·sqrt(2) = 0 :
> eq3 := V1 − F/sqrt(2) − (1 − a2)·Np = 0 :
> eq4 := (1 − a1)·Np − F/sqrt(2) − V2 = 0 :
> eq5 := a2·Mp + V2·8·a·sqrt(2) = 0 :
> solve({eq1, eq2, eq3, eq4, eq5}, {q, a1, a2, V1, V2});
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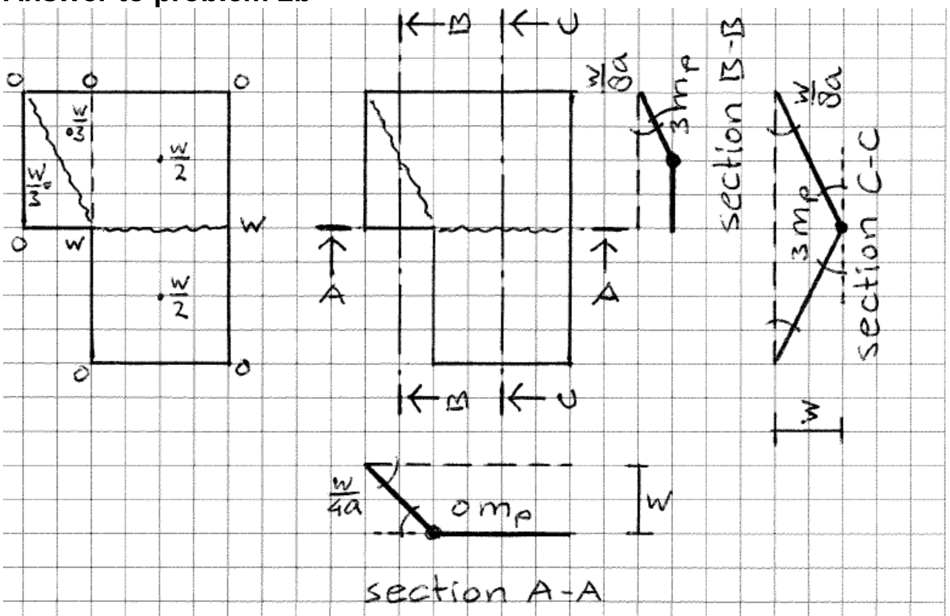
$$\left\{V1 = \frac{0.4027256365\,Mp}{a},\ V2 = -\frac{0.08674543530\,Mp}{a},\ a1 = 0.9988386688,\ a2 = 0.9814125686,\ q = \frac{0.02633168344\,Mp}{a^2}\right\}$$

Answer to problem 2a

D

3 or less correct …….0 point
4 correct ……............ 0.3 point
5 correct ……………..0.7 point
6 correct ……………..1.0 point

Answer to problem 2b

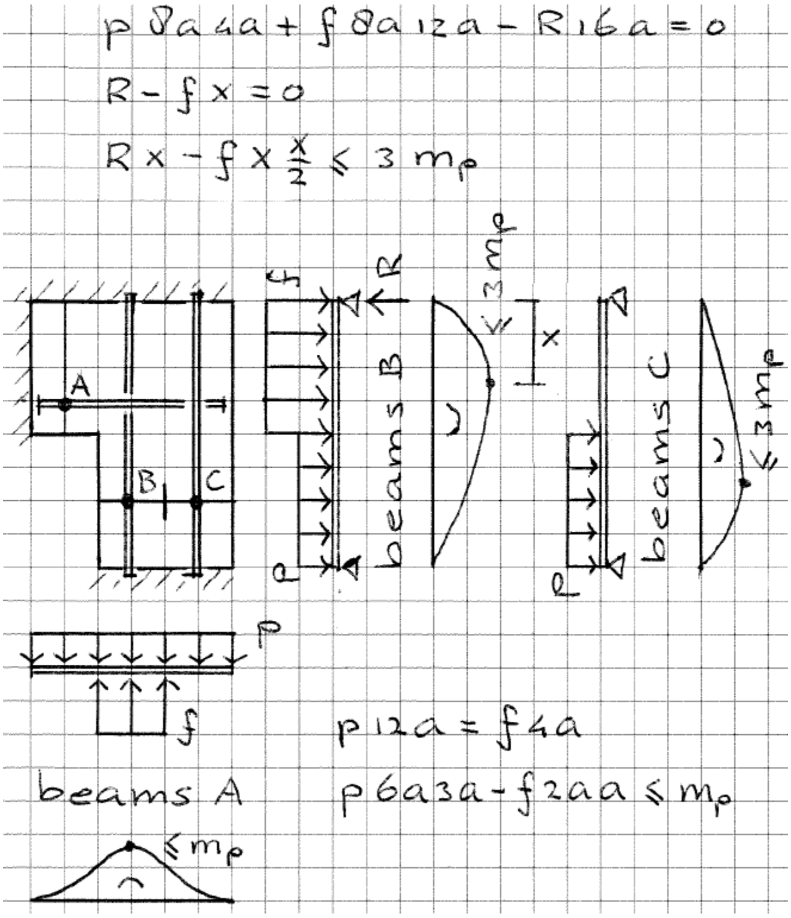


$$E = 3m_p\,8a\left(\frac{w}{8a}+\frac{w}{8a}\right) + 3m_p\,4a\,\frac{w}{8a} + 0 = \frac{15}{2}m_p w$$

$$A = 2\left[p\,8a\,8a\,\frac{w}{2}\right] + 2\left[p\,\frac{1}{2}\,4a\,8a\,\frac{w}{3}\right] = \frac{224}{3}p a^2 w$$

$$E = A \;\Rightarrow\; p = \frac{45}{448}\frac{m_p}{a^2} = 0.100\frac{m_p}{a^2}$$

Answer to problem 2c



The solution to the equations is $p = \frac{9}{200}\frac{m_p}{a^2} = 0.045\frac{m_p}{a^2}$.

Answer to problem 3

a D

b A

c The following answers are correct

- Rankine
- Johansen
- square
- $-m_p < m_2, m_1 < m_p$
- $(m_{px} - m_{xx})(m_{py} - m_{yy}) \geq m_{xy}^2$
 $(m'_{px} + m_{xx})(m'_{py} + m_{yy}) \geq m_{xy}^2$

By comparing to experiments. Although, the latter of the four answers has been derived from equilibrium.