Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Exam CIE4150 Plastic Analysis of Structures
Thursday 7 April 2022, 13:30 – 16:30 hours

Write your name and study number at the top of your work.

Also write whether you were a member of the elastic team, plastic team or no team.

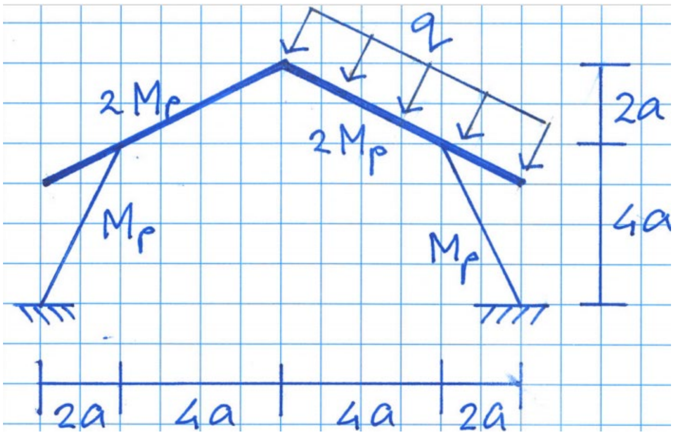


Figure 1. Frame structure

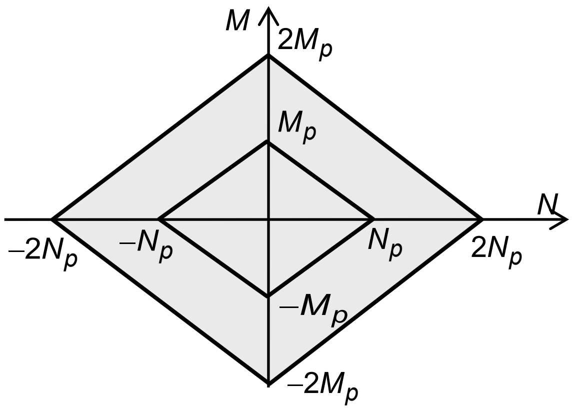


Figure 2. Yield contours

Problem 1

A frame consists of four members (Fig.1). The columns have a strength M_p. The roof members have a strength $2M_p$. All members are rigidly connected. The supports are fixed. The structure is loaded by an evenly distributed line load q per length of roof member (wind load). The relation of Figure 2 exists between the plastic moments and the plastic normal forces.

$$N_p = \beta \frac{M_p}{a}$$

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

a Assume $\beta \to \infty$. Determine the collapse load q for all possible mechanisms. Write the collapse loads as functions of M_p and a. What is the decisive collapse load? (1.5 point)

b Assume $\beta \to \infty$. Draw the bending moment diagram and normal force diagram for the structure at the moment of collapse. (1.5 points)

c Assume β = 7. Choose one of the following problems (You need not do both).
– Determine the largest lower-bound for q.
– Determine the smallest upper-bound for q.
You only need to write down the equations and not solve the equations (1.5 points).

Problem 2

A reinforced concrete plate has simply supported edges and free edges (Fig. 3). It carries an evenly distributed load p [kN/m^2]. There is no other load on the plate. The plate is homogeneous and orthotropic.

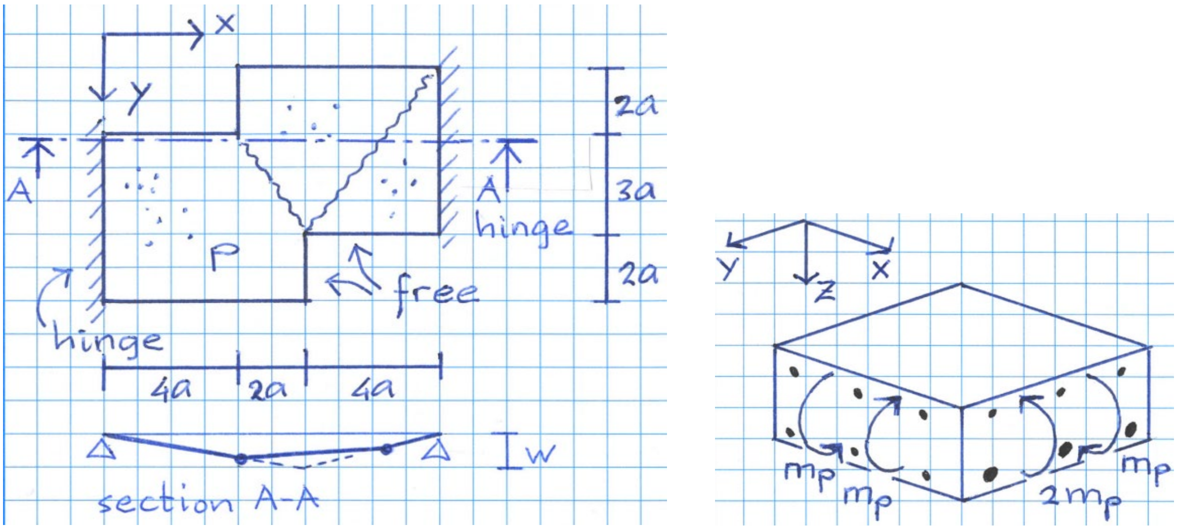


Figure 3. Plate dimensions and reinforcement

a Consider the yield line patterns of Figure 4. Which of these patterns give kinematically possible mechanisms? (1 point)

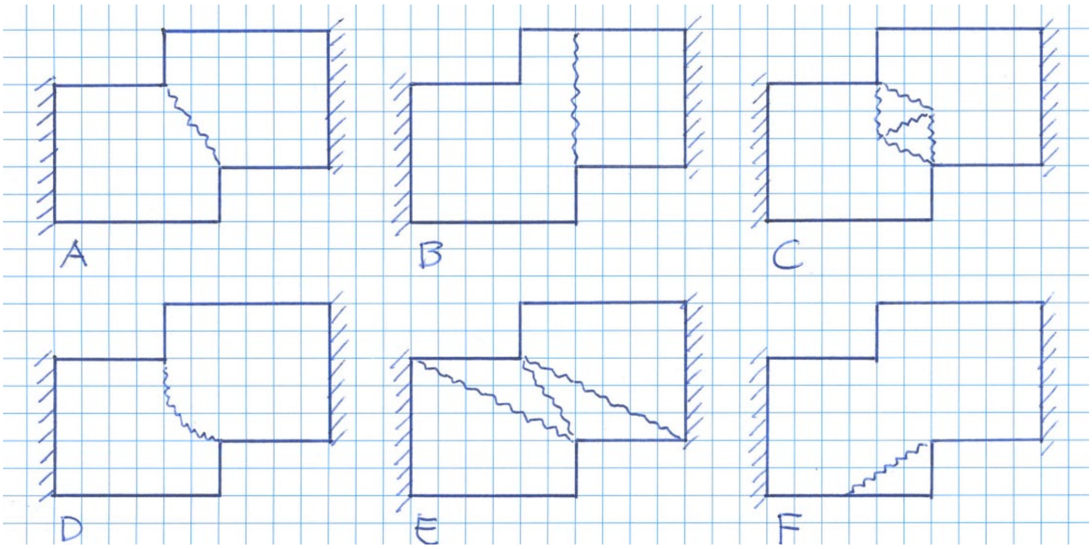


Figure 4. Yield line patterns of problem 2a

b Consider the yield line pattern of Figure 3. Determine an <u>upper bound</u> for p expressed in m_p and a (1.5 point).

c Determine the largest <u>lower-bound</u> for p using torsion free beams ($m_{xy} = 0$). You only need to write down the equations and not solve the equations. (1.5 point)

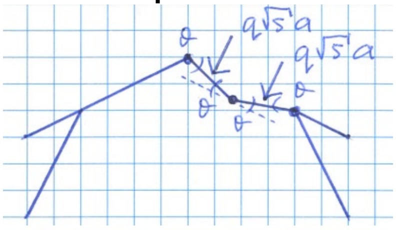
Problem 3

a Which yield criterion is used for stainless steel? (0.5 point)

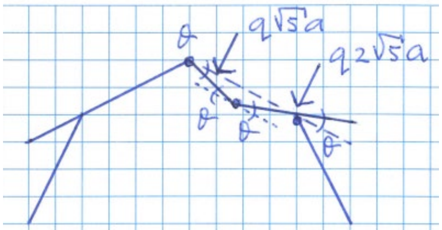
b What is normality of the plastic deformation? (0.5 point)

c What is the relation between mechanism and statically indeterminated? (0.5 point)

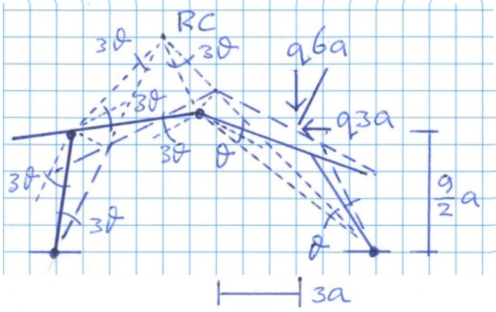
Answer to problem 1a



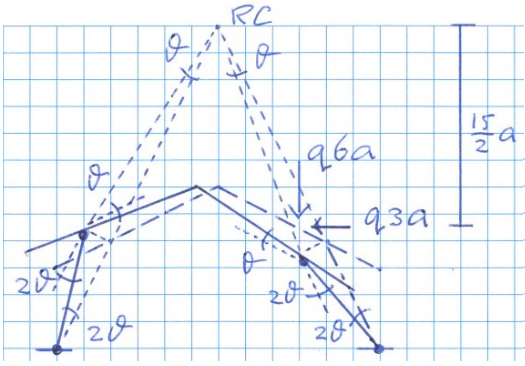
$$E = 2M_p\vartheta + 2M_p(\vartheta+\vartheta) + 2M_p\vartheta$$
$$A = 2\left(q\sqrt{5}a\,\vartheta\,\frac{\sqrt{5}}{2}a\right)$$
$$E = A \;\Rightarrow\; q = \frac{8}{5}\frac{M_p}{a^2}$$



$$E = 2M_p\vartheta + 2M_p(\vartheta+\vartheta) + M_p\vartheta$$
$$A = q\sqrt{5}a\,\vartheta\,\frac{\sqrt{5}}{2}a$$
$$E = A \;\Rightarrow\; q = \frac{14}{5}\frac{M_p}{a^2}$$

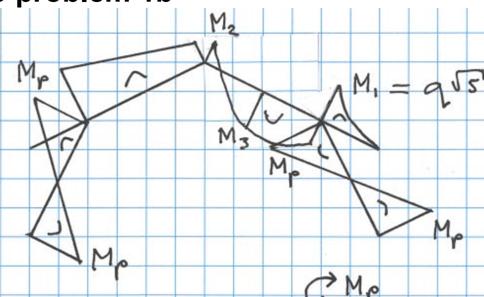


$$E = M_p3\vartheta + M_p(3\vartheta+3\vartheta) + 2M_p(3\vartheta+\vartheta) + M_p\vartheta$$
$$A = q3a\,\vartheta\,\frac{9}{2}a + q6a\,\vartheta\,3a$$
$$E = A \;\Rightarrow\; q = \frac{4}{7}\frac{M_p}{a^2}$$

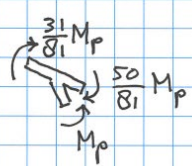


$$E = M_p2\vartheta + M_p(2\vartheta+\vartheta) + M_p(\vartheta+2\vartheta) + M_p2\vartheta$$
$$A = q3a\,\vartheta\,\frac{15}{2}a + q6a\,\vartheta\,3a$$
$$E = A \;\Rightarrow\; \boxed{q = \frac{20}{81}\frac{M_p}{a^2}} \quad \text{decisive}$$

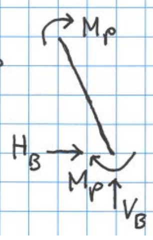
Answer to problem 1b



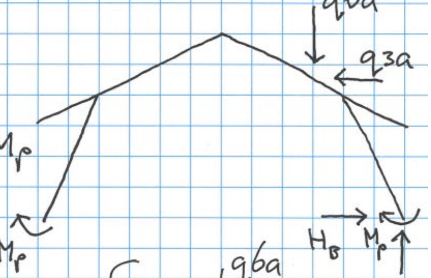
$$M_1 = q\sqrt{5}a\,\frac{\sqrt{5}}{2}a = \frac{50}{81}M_p$$



① $M_p = H_B4a + V_B2a - M_p$

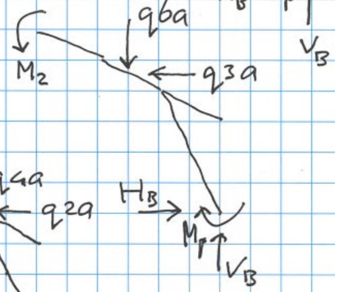


② $M_p = V_B\, 12a - q6a\,9a + q3a\,\frac{9}{2}a - M_p$



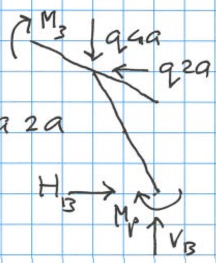
① } $H_B = 0$

② } $V_B = \frac{M_p}{a}$



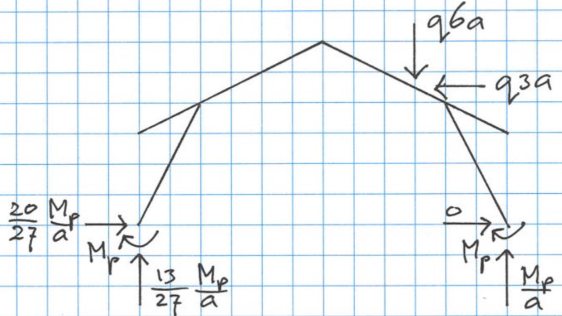
$M_2 = q6a\,3a + q3a\,\frac{3}{2}a - H_B\,6a - V_B\,6a + M_p$

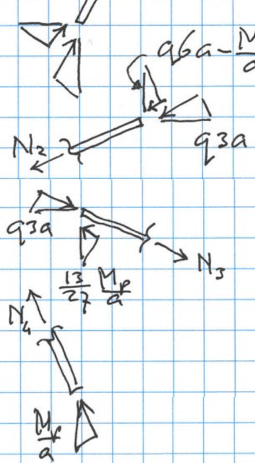
$= \frac{5}{9}M_p$



$M_3 = V_B\,4a + H_B\,5a - M_p - q\,2a\,a - q4a\,2a$

$= \frac{43}{81}M_p$





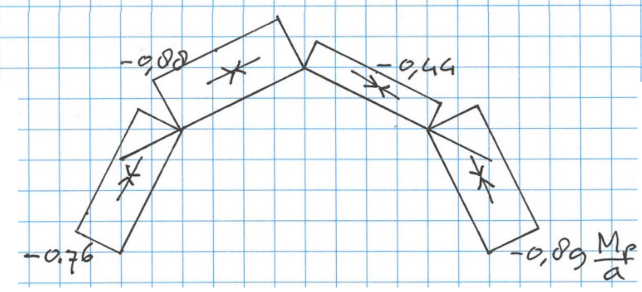
$N_1 = -\left(\frac{13}{27}\frac{M_p}{a}\frac{2}{\sqrt{5}} + \frac{20}{27}\frac{M_p}{a}\frac{1}{\sqrt{5}}\right) = -0.76\,\frac{M_p}{a}$

$q6a - \frac{M_p}{a} = \frac{13}{27}\frac{M_p}{a}$

$N_2 = -\left(q3a\frac{2}{\sqrt{5}} + \frac{13}{27}\frac{M_p}{a}\frac{1}{\sqrt{5}}\right) = -0{,}88\,\frac{M_p}{a}$

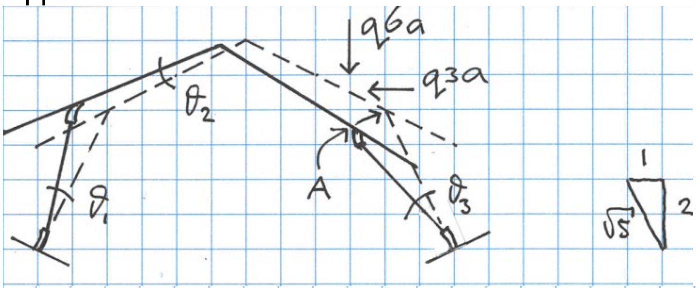
$N_3 = -\left(q3a\frac{2}{\sqrt{5}} - \frac{13}{27}\frac{M_p}{a}\frac{1}{\sqrt{5}}\right) = -0{,}44\,\frac{M_p}{a}$

$N_4 = -\frac{M_p}{a}\frac{2}{\sqrt{5}} = -0{,}89\,\frac{M_p}{a}$



Answer to problem 1c

Upperbound



hor. disp. A ←

$$\vartheta_1 4a + \vartheta_1 \frac{a}{\beta}\frac{1}{\sqrt{5}} + (\vartheta_1+\vartheta_2)\frac{a}{\beta}\frac{1}{\sqrt{5}} = \vartheta_3 4a - \vartheta_3\frac{a}{\beta}\frac{1}{\sqrt{5}} - (\vartheta_2+\vartheta_3)\frac{a}{\beta}\frac{1}{\sqrt{5}}$$

vert. disp. A ↓

$$-\vartheta_1 2a + \vartheta_1\frac{a}{\beta}\frac{2}{\sqrt{5}} + (\vartheta_1+\vartheta_2)\frac{a}{\beta}\frac{2}{\sqrt{5}} + \vartheta_2 8a = \vartheta_3 2a + \vartheta_3\frac{a}{\beta}\frac{2}{\sqrt{5}} + (\vartheta_2+\vartheta_3)\frac{a}{\beta}\frac{2}{\sqrt{5}}$$

$$E = M_p\vartheta_1 + M_p(\vartheta_1+\vartheta_2) + M_p(\vartheta_2+\vartheta_3) + M_p\vartheta_3$$

$$A = q3a\left(\vartheta_3 4a - \vartheta_3\frac{a}{\beta}\frac{1}{\sqrt{5}} - (\vartheta_2+\vartheta_3)\frac{a}{\beta}\frac{1}{\sqrt{5}} - \vartheta_2\frac{1}{2}a\right) +$$
$$+ q6a\left(\vartheta_3 2a + \vartheta_3\frac{a}{\beta}\frac{2}{\sqrt{5}} + (\vartheta_2+\vartheta_3)\frac{a}{\beta}\frac{2}{\sqrt{5}} - \vartheta_2 a\right)$$

$$E = A$$

The solution to the equations is

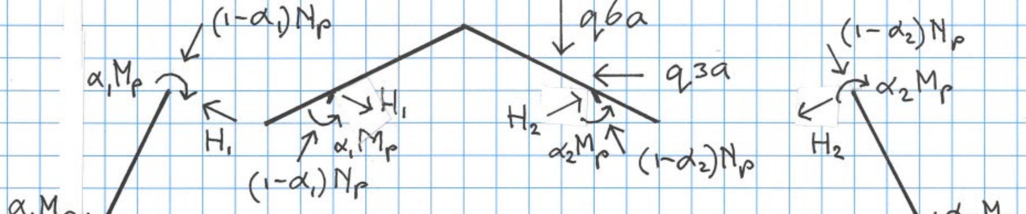
$q = 0{,}2207\,\frac{M_p}{a^2}$ $\quad\vartheta_1 = 1{,}91\,\vartheta_2$ $\quad\vartheta_3 = 2{,}07\,\vartheta_2$

Lowerbound

$$q6a = (1-\alpha_1)N_p\frac{2}{\sqrt{5}} - H_1\frac{1}{\sqrt{5}} + (1-\alpha_2)N_p\frac{2}{\sqrt{5}} + H_2\frac{1}{\sqrt{5}}$$

$$q3a = (1-\alpha_1)N_p\frac{1}{\sqrt{5}} + H_1\frac{2}{\sqrt{5}} - (1-\alpha_2)N_p\frac{1}{\sqrt{5}} + H_2\frac{2}{\sqrt{5}}$$

$$\alpha_1 M_p = q6a\,7a - q3a\frac{a}{2} - \alpha_2 M_p - (1-\alpha_2)N_p\frac{2}{\sqrt{5}}8a - H_2\frac{1}{\sqrt{5}}8a$$



$$N_p = 7\frac{M_p}{a}$$

$\alpha_1 M_p + \alpha_1 M_p = H_1 2\sqrt{5}a$ $\qquad \alpha_2 M_p + \alpha_2 M_p = H_2 2\sqrt{5}a$

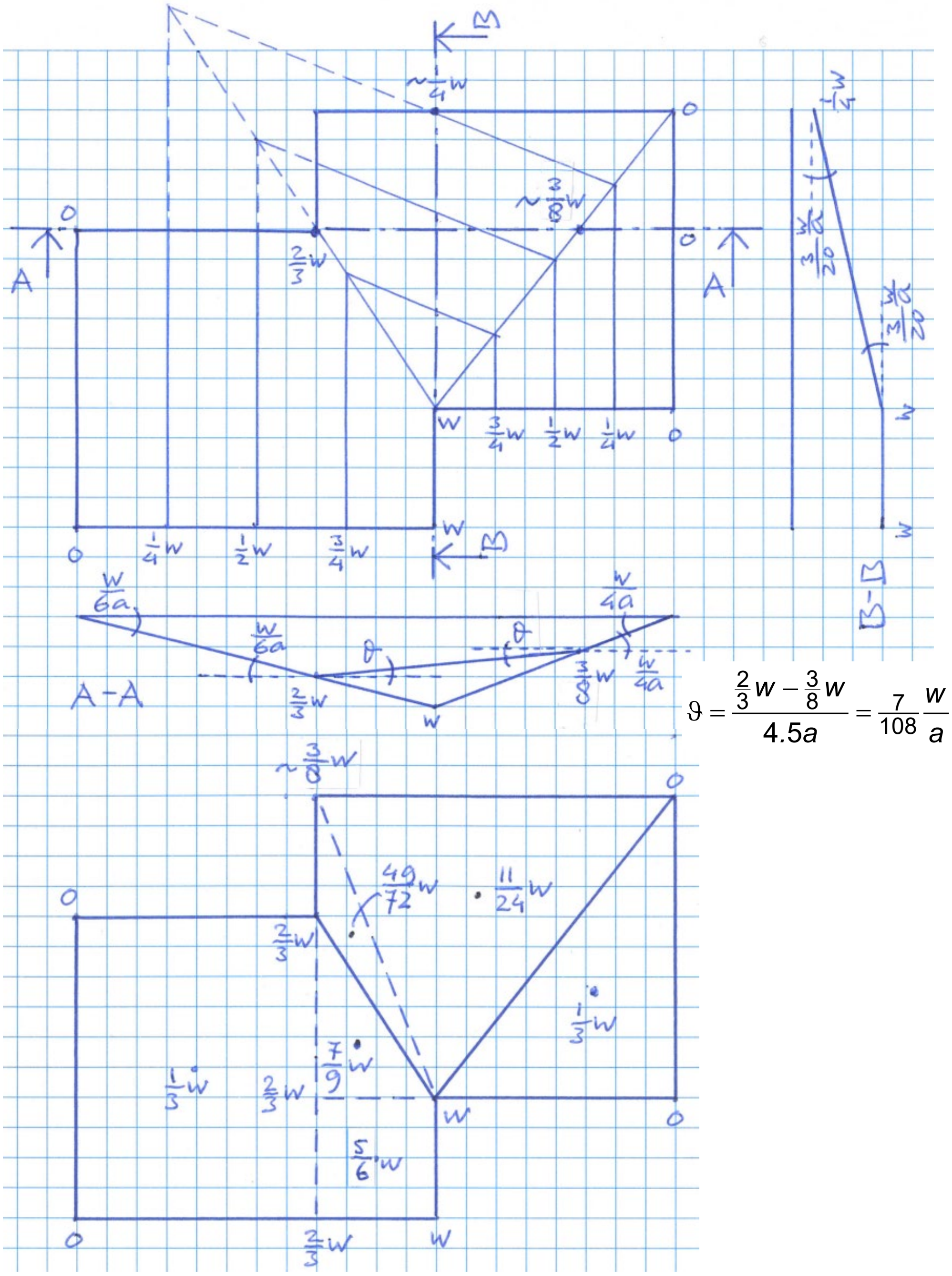
The solution to the equations is

$q = 0{,}2207\,\frac{M_p}{a^2}$ $\quad\alpha_1 = 0{,}90$ $\quad\alpha_2 = 0{,}89$ $\quad H_1 = 0{,}40\,\frac{M_p}{a}$ $\quad H_2 = 0{,}40\,\frac{M_p}{a}$

Answer to problem 2a
B, E, F
(E can move in two ways; so it has too many yield lines)

3 or less correct ……. 0.0 point
4 correct ……………. 0.3 point
5 correct ……………. 0.7 point
6 correct …………….. 1.0 point

Answer to problem 2b



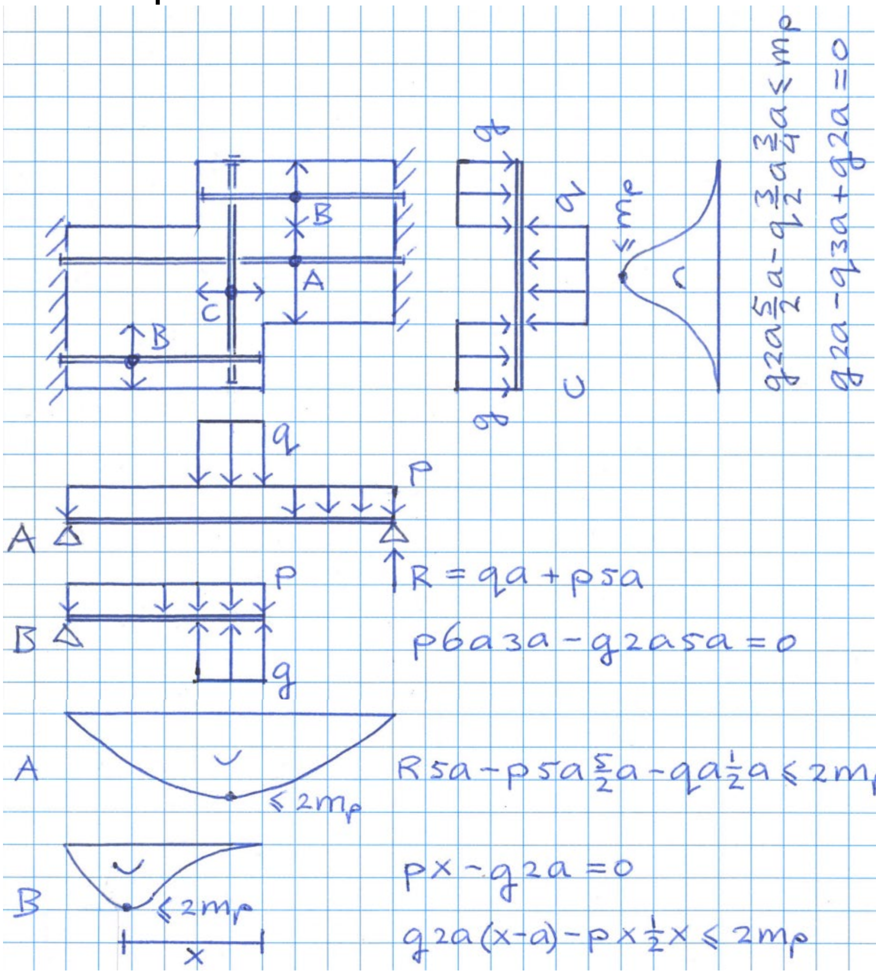
$$\vartheta = \frac{\frac{2}{3}w - \frac{3}{8}w}{4.5a} = \frac{7}{108}\frac{w}{a}$$

$$E = 2m_p 3a\left(\frac{w}{6a} + \frac{7}{108}\frac{w}{a}\right) + m_p 2a \frac{3}{20}\frac{w}{a} + 2m_p 5a\left(\frac{w}{4a} - \frac{7}{108}\frac{w}{a}\right) + m_p 4a \frac{3}{20}\frac{w}{a}$$

$$A = p\,4a5a\frac{1}{3}w + p\,2a2a\frac{5}{6}w + p\frac{2a3a}{2}\frac{7}{9}w + p\frac{4a5a}{2}\frac{1}{3}w +$$
$$+ p\frac{6a5a}{2}\frac{11}{24}w + p\frac{2a2a}{2}\frac{49}{72}w$$

$$E = A \Rightarrow p = 0{,}173\frac{m_p}{a^2}$$

Answer to problem 2c



$$g2a\frac{5}{2}a - q\frac{3}{2}a\frac{3}{4}a \leq m_p$$

$$g2a - q3a + g2a = 0$$

$$R = qa + p5a$$

$$p6a3a - g2a5a = 0$$

$$R5a - p5a\frac{5}{2}a - qa\frac{1}{2}a \leq 2m_p$$

$$px - g2a = 0$$

$$g2a(x-a) - px\frac{1}{2}x \leq 2m_p$$

```
> eq1:= R=q*a+p*5*a:
> eq2:= p*6*a*3*a-g*2*a*5*a=0:
> eq3:= R*5*a-p*5*a*5/2*a-q*a*1/2*a=2*mp:
> eq4:= p*x-g*2*a=0:
> eq5:= g*2*a*(x-a)-p*x*1/2*x<2*mp:
> eq6:= g*2*a*5/2*a-q*3/2*a*3/4*a<mp:
> eq7:= g*2*a-q*3*a+g*2*a=0:
> opl:=solve({eq1,eq2,eq3,eq4,eq7},{R,x,p,g,q}); assign(opl):
```

$$opl := \left\{R = \frac{148\,mp}{233\,a},\ g = \frac{36\,mp}{233\,a^2},\ \boxed{p = \frac{20\,mp}{233\,a^2}},\ q = \frac{48\,mp}{233\,a^2},\ x = \frac{18\,a}{5}\right\}$$

```
> evalf(eq5);
```

$$0.2472103004\,mp < 2.\,mp$$

```
> evalf(eq6);
```

$$0.5407725322\,mp < mp$$

Answer to problem 3

a Von Mises

b The vector perpendicular to the yield surface gives the plastic deformations.

c The number of plastic hinges required for a *mechanism* is less than or equal to the degree of *statically indeterminacy* plus one.

grades