Delft University of Technology
Faculty of Civil Engineering and Geosciences
Structural Mechanics Section

Write your name and study number at the top of your work.

Exam CIE4150 Plastic Analysis of Structures
Thursday 13 April 2023, 13:30 – 16:30 hours

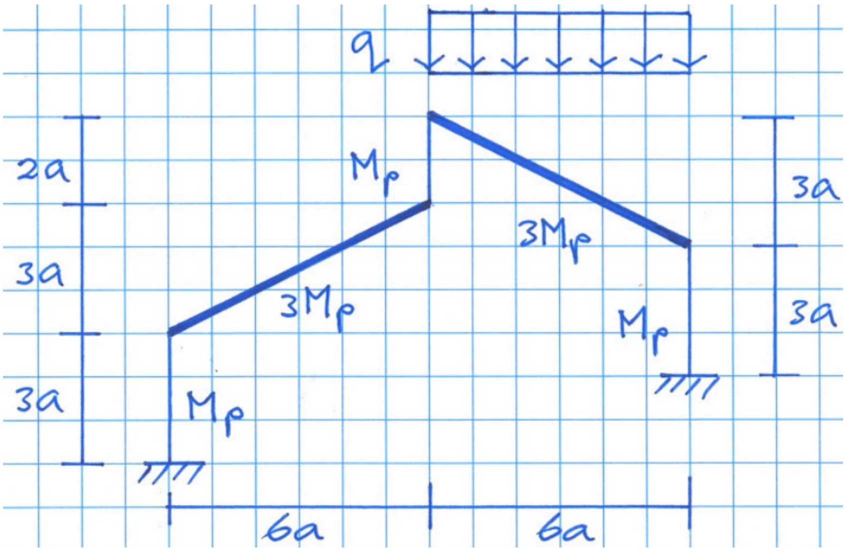


Figure 1. Frame structure

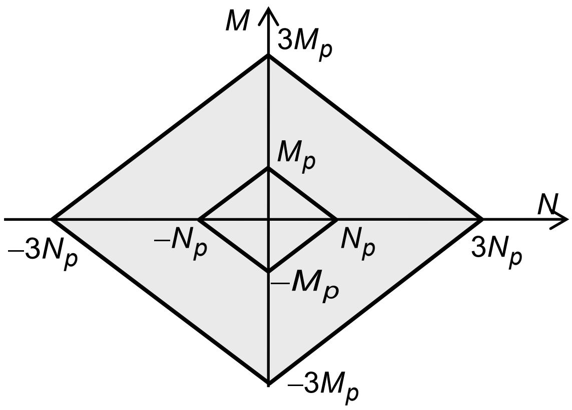


Figure 2. Yield contours

Problem 1

A frame consists of five members (Fig.1). The columns have a strength M_p. The roof members have a strength $3M_p$. The members are rigidly connected. The supports are fixed. The structure is loaded by an evenly distributed line load q per horizontal width (snow load). The relation of Figure 2 exists between the plastic moments and the plastic normal forces.

$$N_p = \beta \frac{M_p}{a}$$

The influence of shear on the yield contour is neglected. Buckling and second order effects are not considered.

a Assume $\beta \to \infty$. Determine the collapse load q for all possible mechanisms. Write the collapse loads as functions of M_p and a. What is the decisive collapse load? (1.5 point)

b Assume $\beta \to \infty$. Draw the bending moment diagram and normal force diagram for the structure at the moment of collapse. (1.5 points)

c Assume β = 7. Choose one of the following problems (You need not do both).
– Determine the largest lower-bound for q.
– Determine the smallest upper-bound for q.
You only need to write down the equations and not solve the equations (1.5 points).

Problem 2

A reinforced concrete plate has simply supported edges and free edges (Fig. 3). It carries an evenly distributed load p [kN/m^2]. There is no other load on the plate. The plate is homogeneous and orthotropic.

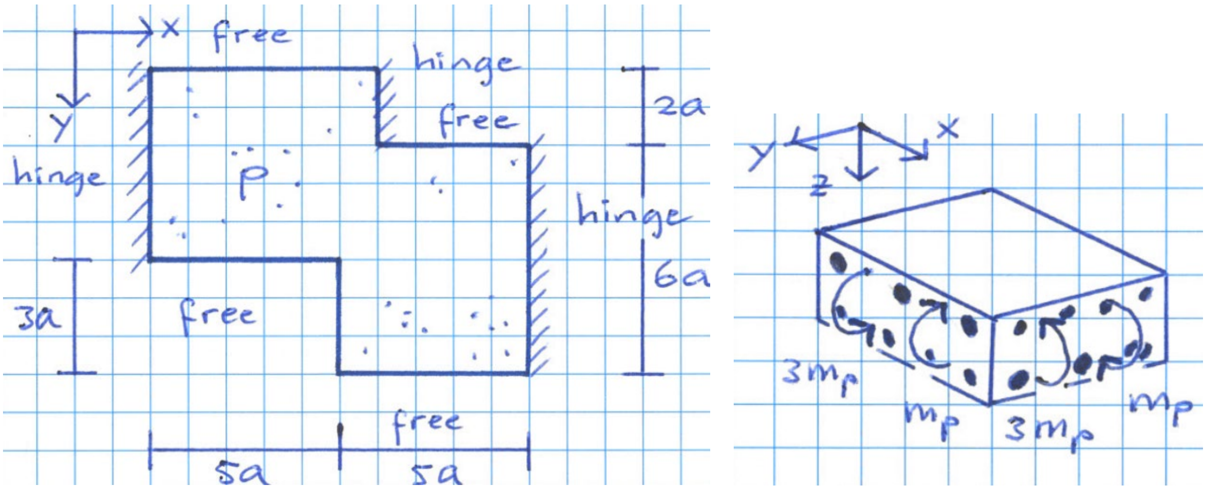


Figure 3. Plate dimensions and reinforcement

a Consider the yield line patterns of Figure 4. Which of these patterns give kinematically possible mechanisms? (1 point)

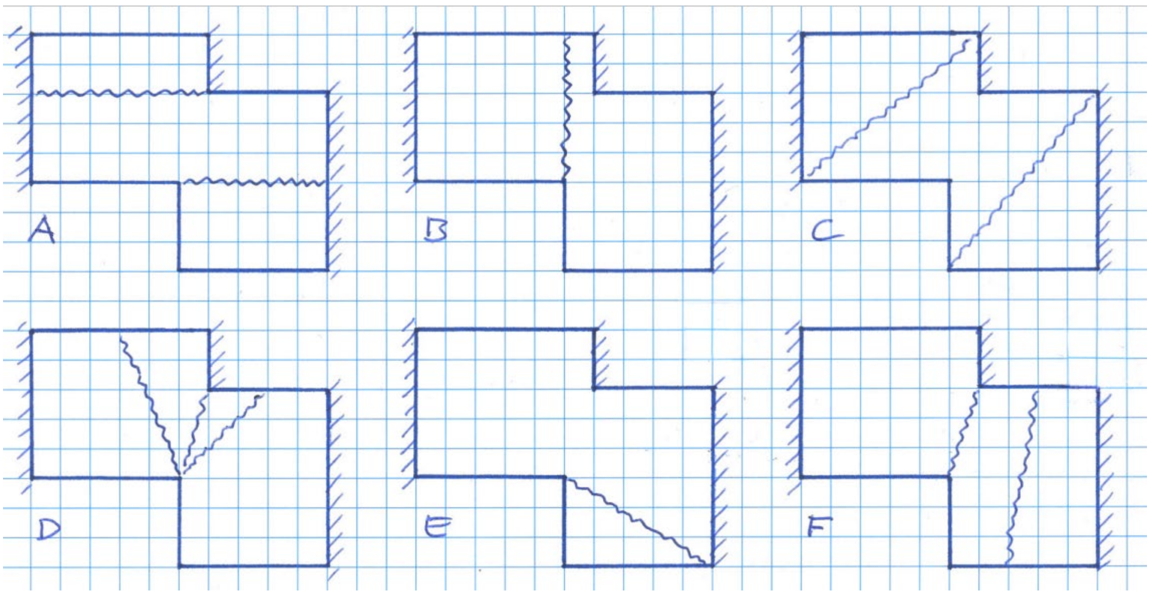


Figure 4. Yield line patterns of problem 2a

b Consider the yield line pattern of Figure 5. Determine an <u>upper-bound</u> for p expressed in m_p and a (1.5 point).

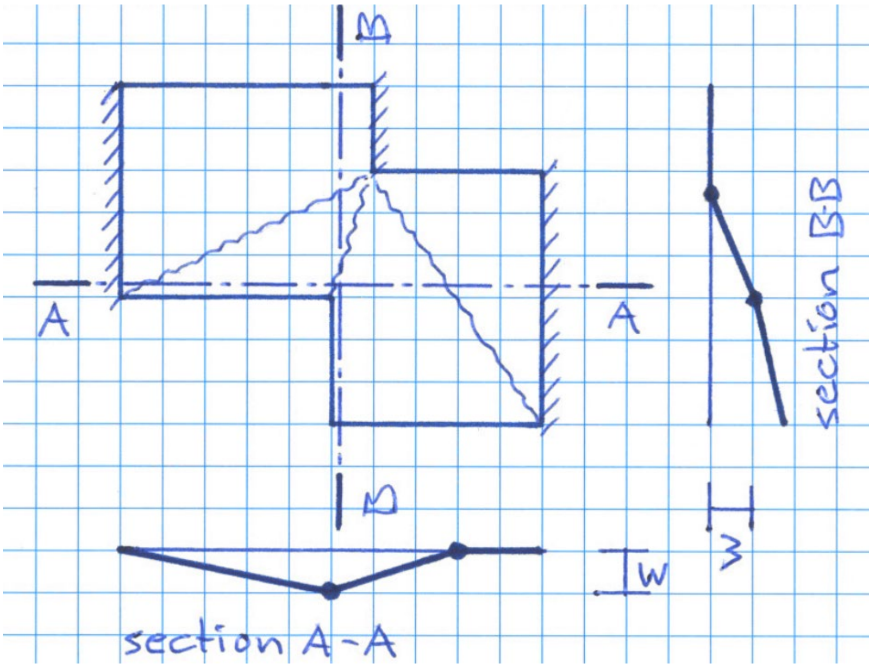


Figure 5. Mechanism of Problem 2b

c Determine the largest <u>lower-bound</u> for p using torsion free beams ($m_{xy} = 0$). You need only to write down the equations and not solve the equations. (1.5 point)

Also see the other side of this page.

Problem 3

a How is it that lower-bounds are smaller than or equal to upper-bounds?
Choose A, B, C or D. (0.5 point)

A Torsion is neglected.
B They include the elastic solution.
C There is a mathematical proof of this.
D They are intended to be a safe approximation of the exact plastic solution.

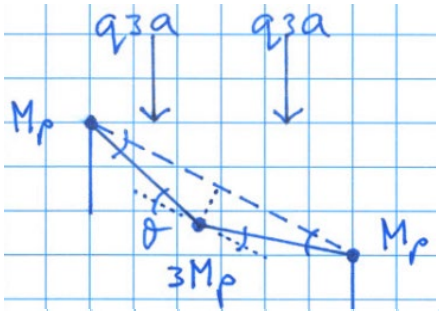
b Suppose we do a upper-bound analysis and determine the decisive mechanism. We plot the moment distribution and observe that a moment somewhere is larger than the moment capacity. What do we conclude? Choose A, B, C or D. (0.5 point)

A The structure does not fulfil the design requirements.
B An even more decisive mechanism exists.
C We made a calculation error.
D No problem; this can happen in upper-bound analysis.

c In general; how do we know we found the exact plastic collapse load?
Choose A, B, C or D. (0.5 point)

A The upper-bound and lower-bound are the same.
B We have tried all possible mechanisms and picked the smallest.
C By comparing to a nonlinear finite element analysis.
D After checking by a fellow student or colleague.

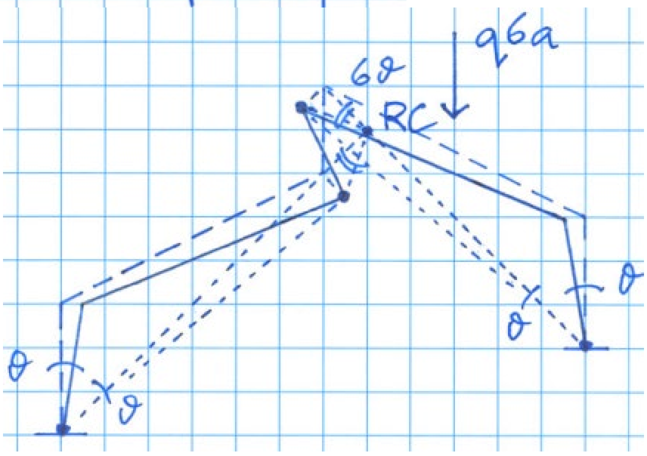
Answer to problem 1a




```
> restart:
> E:= Mp*t + 3*Mp*(t+t) + Mp*t:
> A:= 2*(q*3*a * t * 3*a/2):
> q:= solve(E=A,q); evalf(q);
```

$$q := \frac{8\,Mp}{9\,a^2}$$

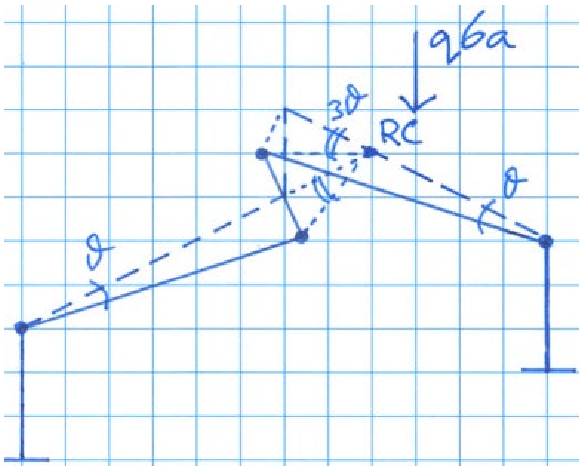
$$\frac{0.8888888889\,Mp}{a^2}$$




```
> restart:
> E:= Mp*t + Mp*(t+6*t) + Mp*(6*t-t) + Mp*t:
> A:= q*6*a * t * 3*a:
> q:= solve(E=A,q); evalf(q);
```

$$q := \frac{7\,Mp}{9\,a^2}$$

$$\frac{0.7777777778\,Mp}{a^2}$$

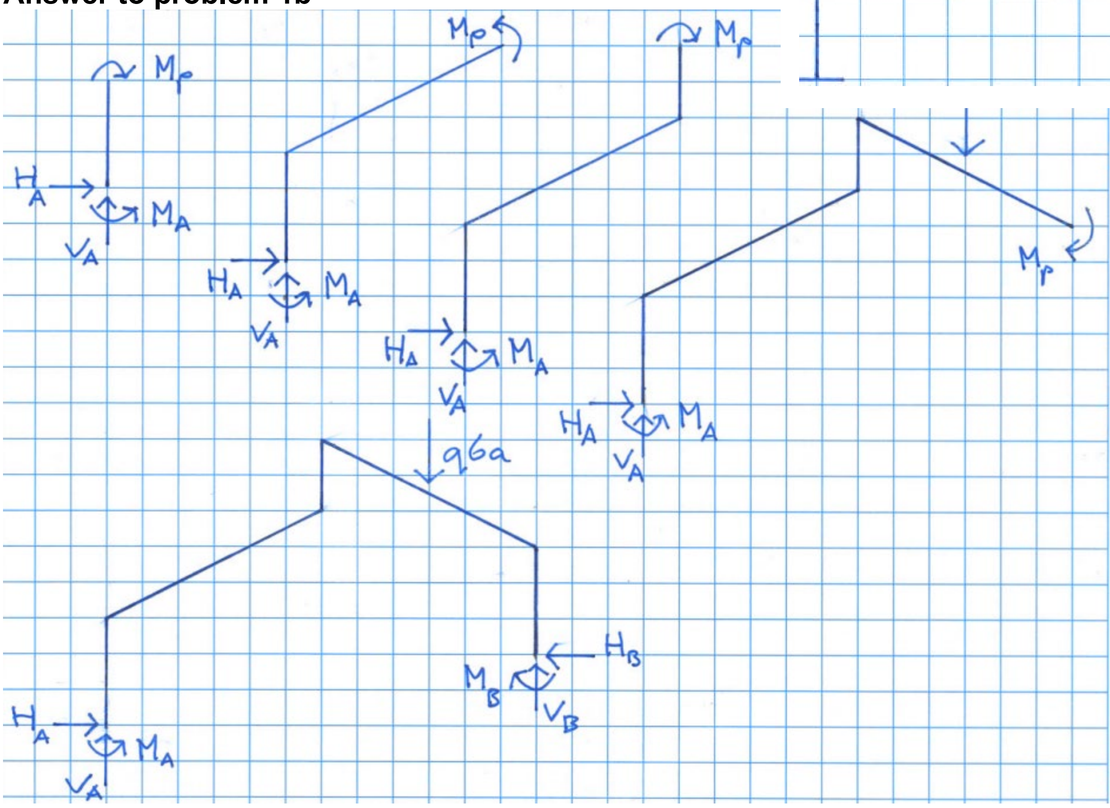



```
> restart:
> E:= Mp*t + Mp*(t+3*t) + Mp*(3*t-t) + Mp*t:
> A:= q*6*a * t * 3*a:
> q:= solve(E=A,q); evalf(q);
```

decisive $\boxed{q := \frac{4\,Mp}{9\,a^2}}$

$$\frac{0.4444444444\,Mp}{a^2}$$

Answer to problem 1b



```
> q:=4/9*Mp/a^2:
> l:=sqrt((6*a)^2+(3*a)^2):
> MC:=1/8*q*(6*a)^2-Mp;
```

$$MC := Mp$$

```
> eq1:= Mp=HA*3*a+MA:
> eq2:= Mp=VA*6*a-HA*6*a-MA:
> eq3:= Mp=HA*8*a-VA*6*a+MA:
> opl:=solve({eq1,eq2,eq3},{HA,VA,MA}); assign(opl):
```

$$opl := \left\{HA = \frac{Mp}{a}, MA = -2\,Mp, VA = \frac{5\,Mp}{6\,a}\right\}$$

```
> check:= Mp=HA*5*a-VA*12*a+MA+q*6*a*3*a;
```

$$check := Mp = Mp$$

```
> MB:=HA*2*a-VA*12*a+MA+q*6*a*3*a;
```

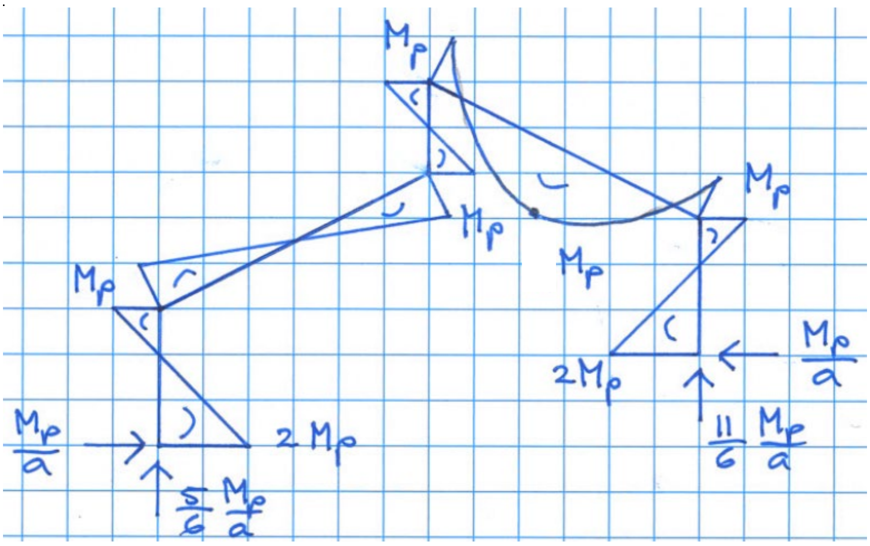
$$MB := -2\,Mp$$

```
> VB:=q*6*a-VA;
```

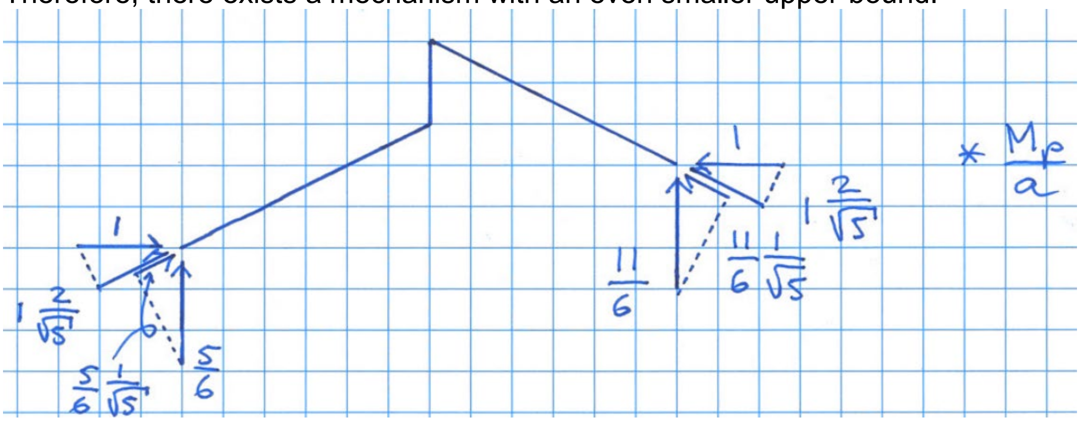
$$VB := \frac{11\,Mp}{6\,a}$$

```
> HB:=HA;
```

$$HB := \frac{Mp}{a}$$



The moment is larger than the moment capacity at two places. Therefore, the present solution does not fulfil the lower-bound criteria. Therefore, the exact solution has not been found. Therefore, there exists a mechanism with an even smaller upper-bound.



```
> N2:=HA*2/sqrt(5) + VA*1/sqrt(5); evalf(N2);
```

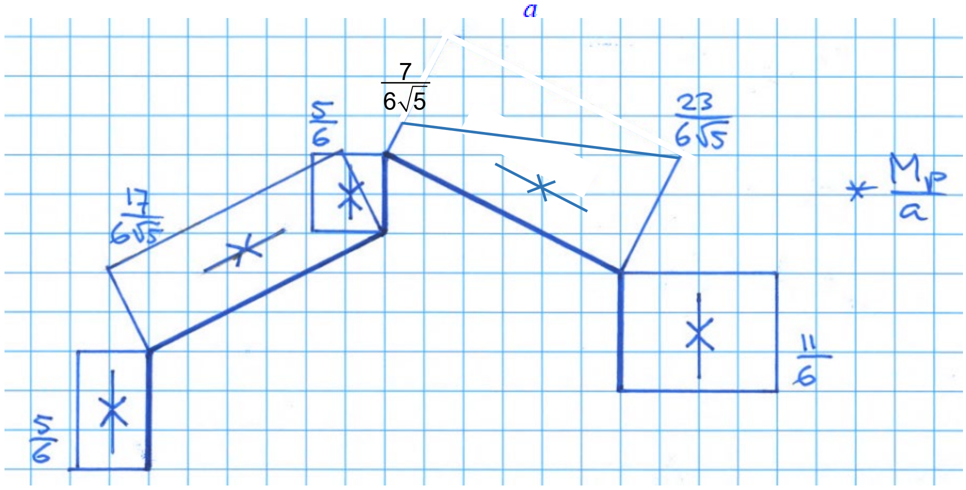
$$N2 := \frac{17\,Mp\,\sqrt{5}}{30\,a}$$

$$\frac{1.267105187\,Mp}{a}$$

```
> N4:=HB*2/sqrt(5) + VB*1/sqrt(5); evalf(N4);
```

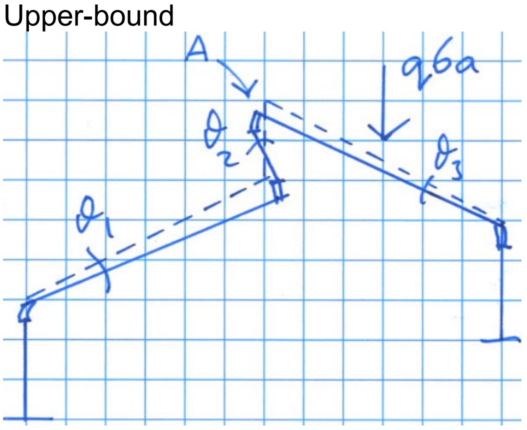
$$N4 := \frac{23\,Mp\,\sqrt{5}}{30\,a}$$

$$\frac{1.714318782\,Mp}{a}$$



Answer to problem 1c

Upper-bound

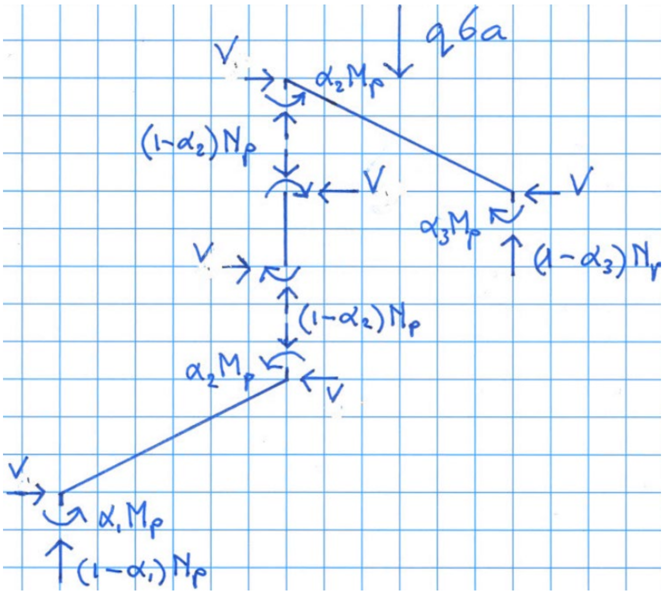



```
> b:=7:
> eq1:= t1*3*a - t2*2*a = -t3*3*a:
> eq2:= t1*a/b + t1*6*a +(t1+t2)*a/b + (t2-t3)*a/b = t3*a/b + t3*6*a:
> E:= Mp*t1 + Mp*(t1+t2) + Mp*(t2-t3) + Mp*t3:
> A:= q*6*a * (t3*a/b + t3*3*a):
> opl:=solve({eq1,eq2,E=A},{q,t2,t3}); evalf(opl);
```

$$opl := \left\{ q = \frac{1211\,Mp}{3102\,a^2},\ t2 = \frac{132\,t1}{41},\ t3 = \frac{47\,t1}{41} \right\}$$

$$\left\{ q = \frac{0.3903932946\,Mp}{a^2},\ t2 = 3.219512195\,t1,\ t3 = 1.146341463\,t1 \right\}$$

Lower-bound




```
> b:=7: Np:=b*Mp/a:
> eq1:= (1-a1)*Np = (1-a2)*Np:
> eq2:= a1*Mp = (1-a2)*Np*6*a - a2*Mp - V*3*a:
> eq3:= a2*Mp = V*2*a - a2*Mp:
> eq4:= q*6*a = (1-a2)*Np + (1-a3)*Np:
> eq5:= a2*Mp = q*6*a*3*a + V*3*a - (1-a3)*Np*6*a + a3*Mp:
> op1:=solve({eq1,eq2,eq3,eq4,eq5},{q,V,a1,a2,a3}); evalf(op1);
```

$$opl := \left\{V = \frac{42\ Mp}{47\ a},\ a1 = \frac{42}{47},\ a2 = \frac{42}{47},\ a3 = \frac{399}{517},\ q = \frac{1211\ Mp}{3102\ a^2}\right\}$$

$$\left\{V = \frac{0.8936170213\ Mp}{a},\ a1 = 0.8936170213,\ a2 = 0.8936170213,\ a3 = 0.7717601547,\ q = \frac{0.3903932946\ Mp}{a^2}\right\}$$

Answer to problem 2a

E, F

3 or less correct …….	0.0 point
4 correct …………….	0.3 point
5 correct …………….	0.7 point
6 correct ……………..	1.0 point

Answer to problem 2b

```
> E1:=    mp * 3*a *  w/(5*a)            + 3*mp * 6*a *  w/(5/2*a):
> E2:= 3*mp * 3*a * (w/(5*a)+w/(3*a)) +    mp *    a * (w/(5/2*a)-(4/3*w)/(6*a)):
> E3:=    mp * 6*a *           w/(3*a)   + 3*mp * 4*a *              (4/3*w)/(6*a):
> E:=E1+E2+E3;
```

$$E := \frac{157\ mp\ w}{9}$$

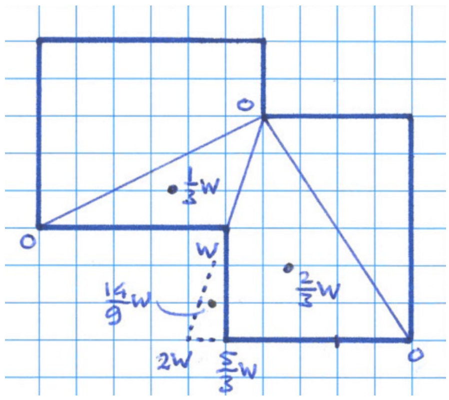
```
> A:= p * (5*a*3*a)/2 * w/3 + p * (6*a*6*a)/2 * 2/3*w - p * (a*3*a)/2 * 14/9*w;
```

$$A := \frac{73\ p\ a^2\ w}{6}$$

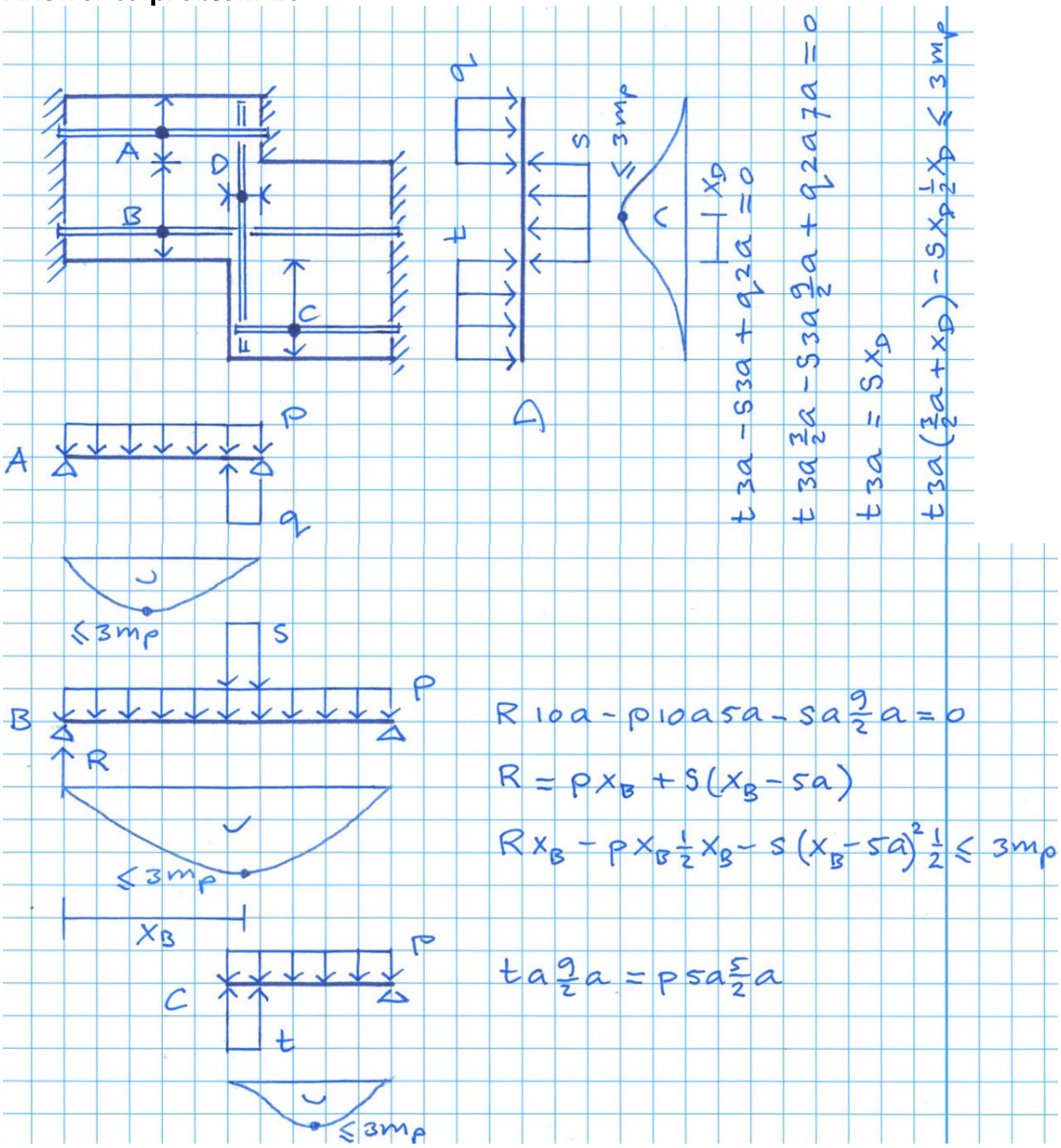
```
> p:=solve(E=A,p); evalf(p);
```

$$p := \frac{314\ mp}{219\ a^2}$$

$$\frac{1.433789954\ mp}{a^2}$$



Answer to problem 2c




```
> eq1:= R*10.*a - p*10*a*5*a - s*a*9/2*a = 0:
> eq2:= R = p*xB + s*(xB-5*a):
> eq3:= R*xB - p*xB*1/2*xB - s*(xB-5*a)^2 = 3*mp:
> eq4:= t*a*9/2*a = p*5*a*5/2*a:
> eq5:= t*3*a - s*3*a + q*2*a = 0:
> eq6:= t*3*a*3/2*a - s*3*a*9/2*a + q*2*a*7*a = 0:
> eq7:= t*3*a = s*xD:
> eq8:= t*3*a*(3/2*a+xD) - s*xD*1/2*xD < 3*mp:
> opl:=solve({eq1,eq2,eq3,eq4,eq5,eq6,eq7},{R,xB,xD,q,s,t,p}); assign(opl):
```

$$opl := \left\{R = \frac{0.8831984086\,mp}{a}, \boxed{p = \frac{0.1139610850\,mp}{a^2}}, q = \frac{0.5698054249\,mp}{a^2}, s = \frac{0.6964288526\,mp}{a^2}, t = \frac{0.3165585694\,mp}{a^2}, xB = 5.386718750\,a, xD = 1.363636364\,a\right\}$$

```
> evalf(eq8);
```

$$2.072019727\,mp < 3.\,mp$$

Answer to problem 3

a C

b B or C

c A