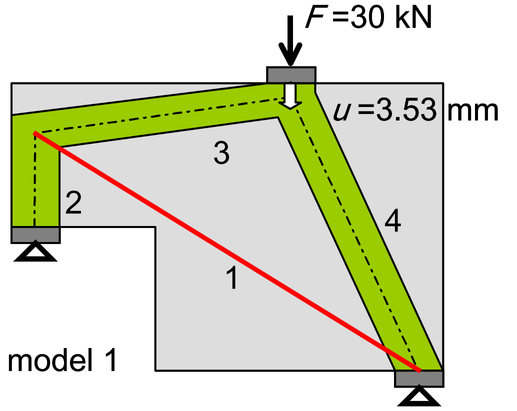
This text is copied from an exam (4 Nov. 2005) of the course Theory of Elasticity.
(At present this course is not available at Delft University)

Problem 3 (3 points)

A structural engineer designs a reinforced concrete deep beam using the strut-and-tie method (Dutch; vakwerkanalogie). He comes up with four equilibrium systems to carry the load *F* (Fig. 4, 5, 6, 7). The force flow and deformation are computed by a frame analysis program. In the analysis the struts have a stiffness *EA* that is thirty times larger than the ties.

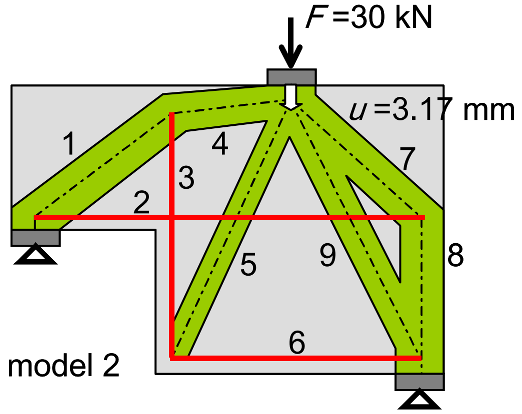
a Check the displacement of the first model using minimum complementary energy or Castigliano's theorem.

b Which of these strut-and-tie models is the best? Explain your answer.



Member i	Axial Force N_i	Length l_i	Stiffness EA_i
1	0.517 F	5.97 m	15000 kN
2	-0.333 F	1.20	450000
3	-0.446 F	3.44	450000
4	-1.034 F	3.98	450000

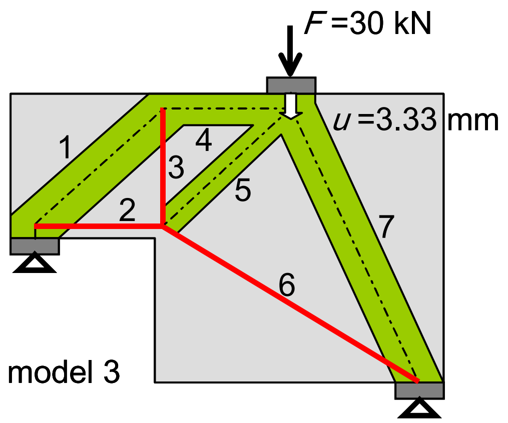
Figure 4. Strut-and-tie model 1



Member i	Axial Force N_i	Length l_i	Stiffness EA_i
1	-0.564 F	2.28 m	450000 kN
2	0.445 F	5.20	15000
3	0.318 F	3.30	15000

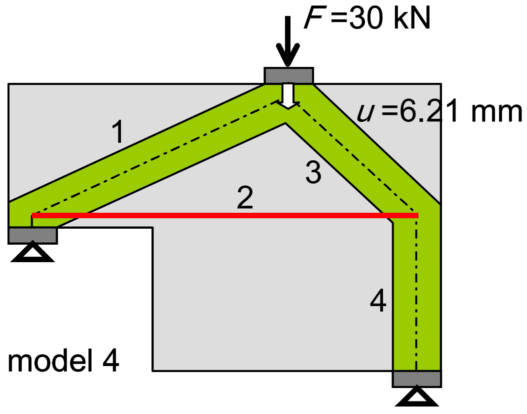
4	-0.446 *F*	1.60	450000
5	-0.352 *F*	3.76	450000
6	0.150 *F*	3.40	15000
7	-0.579 *F*	2.34	450000
8	-0.371 *F*	1.90	450000
9	-0.320 *F*	3.85	450000

Figure 5. Strut-and-tie model 2



Member i	Axial Force N_i	Length l_i	Stiffness EA_i
1	-0.504 *F*	2.27 m	450000 kN
2	0.378 *F*	1.70	15000
3	0.333 *F*	1.50	15000
4	-0.378 *F*	1.70	450000
5	-0.089 *F*	2.27	450000
6	0.522 *F*	4.00	15000
7	-1.041 *F*	3.98	450000

Figure 6. Strut-and-tie model 3



Member i	Axial Force N_i	Length l_i	Stiffness EA_i
1	-0.826 *F*	3.72 m	450000 kN
2	0.756 *F*	5.10	15000
3	-1.008 *F*	2.27	450000
4	-0.667 *F*	2.10	450000

Figure 7. Strut-and-tie model 4

Answers to Problem 3

a Castigliano

$$E_c = \tfrac{1}{2}\sum_i \frac{N_i^2 l_i}{EA_i} = \tfrac{1}{2}\left(\frac{(0.517\,F)^2 5.97}{15000} + \frac{(-0.333\,F)^2 1.20}{450000} + \frac{(-0.446\,F)^2 3.44}{450000} + \frac{(-1.034\,F)^2 3.98}{450000}\right)$$

$$E_c = 0.0000588\,F^2 \text{ m/kN}$$

$u = \dfrac{\partial E_c}{\partial F} = 2 \times 0.0000588\,F = 2 \times 0.0000588 \times 30 = 0.00353$ m, which is correct.

b The best

Any of the following four answers is correct.

A Each strut-and-tie model is equally suitable. Plasticity theory states that any equilibrium system that does not violate the yield conditions gives a lower bound for the strength of the structure. Experiments have shown that reinforced concrete is sufficiently ductile to apply plasticity theory.

B Strut-and-tie model 1 is the best because the value $\sum N_j l_j$ is the smallest, where j obtains the number of all ties. This leads to the least amount of reinforcement.

Model 1: $\sum N_j l_j$ = 93 kNm

Model 2: $\sum N_j l_j$ = 116 kNm

Model 3: $\sum N_j l_j$ = 97 kNm

Model 4: $\sum N_j l_j$ = 116 kNm

C Strut-and-tie model 2 is the best because its complementary energy is the smallest.

$E_{\text{compl}} = \tfrac{1}{2}\sum \dfrac{N_i^2 l_i}{EA_i}$ where i obtains the numbers of all struts and all ties.

Model 1: E_{compl} = 0.053 kNm

Model 2: E_{compl} = 0.046 kNm

Model 3: E_{compl} = 0.050 kNm

Model 4: E_{compl} = 0.093 kNm

D Strut-and-tie model 2 is the best because its deformation is the smallest. After all, the deformation calculated with minimum complementary energy decreases when the model is improved.

Note that that we cannot use potential energy in this problem because we are not looking for the correct displacements. We use complementary energy because we are looking for the correct force flow.

Note that if you calculate the complementary energy as $E_{\text{compl}} = \tfrac{1}{2}\sum \dfrac{N_i^2 l_i}{EA_i} - Fu$ you assume that the displacement u is imposed. Clearly, not u is imposed but F is imposed.