C.7 Virtual work

Let there be considered a body in equilibrium. If the volume of the body is V the equilibrium conditions are that in the volume V the following equations are satisfied,

$$\sigma_{ij,i} + F_j = 0, \tag{C.26}$$

and

$$\sigma_{ij} = \sigma_{ji}, \tag{C.27}$$

where F_j is a given volume force. The comma indicates partial differentiation,

$$a_{,i} = \frac{\partial a}{\partial x_i}. \tag{C.28}$$

It is assumed that the boundary conditions are that on a part (S_1) of the boundary the stresses are prescribed, and that on the remaining part of the boundary (S_2) the displacements are prescribed,

$$\text{on } S_1 \quad : \quad \sigma_{ij} n_i = t_j, \tag{C.29}$$

$$\text{on } S_2 \quad : \quad u_i = f_i, \tag{C.30}$$

where t_j is given on S_1 and f_i is given on S_2.

In the sequel the following definitions are needed. A field of stresses that satisfies equations (C.26), (C.27) and (C.29) is a *statically admissible* stress field, or an *equilibrium system*. A field of displacements that satisfies certain regularity conditions (meaning that the material should retain its integrity, and that no overlaps or gaps may be created in the deformation, but that allows sliding of one part with respect to the rest of the body), and that satisfies equation (C.30), is a *kinematically admissible* displacement field, or a *mechanism*. To such a field a displacement field can be associated by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{C.31}$$

Now consider an arbitrary statically admissible stress field σ_{ij}, and an arbitrary kinematically admissible displacement field u_i. These fields need not have any relation, except that they must be defined in the same volume V. In general one may write

$$\int_V \sigma_{ij,i} u_j \, dV = \int_V [(\sigma_{ij} u_j)_{,i} - \sigma_{ij} u_{j,i}] \, dV.$$

Using Gauss' divergence theorem and eq. (C.27) it follows that

$$\int_V \sigma_{ij,i} u_j \, dV = \int_S \sigma_{ij} u_j n_i \, dS - \int_V \tfrac{1}{2} \sigma_{ij} (u_{i,j} + u_{j,i}) \, dV.$$

With (C.31), (C.29), (C.30) and (C.27) it now follows that

$$\int_V \sigma_{ij} \varepsilon_{ij} \, dV = \int_{S_1} t_i u_i \, dS + \int_{S_2} \sigma_{ij} n_i f_j \, dS + \int_V F_i u_i \, dV. \tag{C.32}$$

Equation (C.32) is valid for any combination of an arbitrary statically admissible field and an arbitrary kinematically admissible displacement field, defined in the same body.

Equation (C.32) must also be valid for the combination of the statically admissible stress field σ_{ij} and the kinematically admissible displacement field $u_i + \dot{u}_i \, dt$. Because this field should also satisfy the boundary condition (C.30), in order to be kinematically admissible, it follows that

$$\text{on } S_2 : \quad \dot{u}_i = 0. \tag{C.33}$$

The small increments of the displacement field $\dot{u}_i \, dt$, that satisfies (C.33) constitutes a *virtual displacement*. Similar to eq. (C.32) the following equation must be satisfied

$$\int_V \sigma_{ij} \varepsilon_{ij} \, dV + dt \int_V \sigma_{ij} \dot{\varepsilon}_{ij} \, dV = \int_{S_1} t_i u_i \, dS + dt \int_{S_1} t_i \dot{u}_i \, dS + \int_{S_2} \sigma_{ij} n_i f_j \, dS + \int_V F_i u_i \, dV + dt \int_V F_i \dot{u}_i \, dV. \tag{C.34}$$

If eq. (C.32) is subtracted from this equation, the result is, after division by dt,

$$\int_V \sigma_{ij} \dot{\varepsilon}_{ij} \, dV = \int_{S_1} t_i \dot{u}_i \, dS + \int_V F_i \dot{u}_i \, dV. \tag{C.35}$$

This is the *virtual work theorem*. It is valid for any combination of a statically admissible stress field, and a variation of a kinematically admissible displacement field. These fields need not be related at all.

The integral in the left hand side is the (virtual) work by the stresses on the given incremental deformations. The terms in the right hand side can be considered as the (virtual) work by the volume forces and the surface load during the virtual displacement. This virtual work appears to be equal to the work done by the stresses on the incremental strains.